

SCHEMIC GROTHENDIECK RINGS AND MOTIVIC RATIONALITY

HANS SCHOUTENS

ABSTRACT. We propose a suitable substitute for the classical Grothendieck ring of an algebraically closed field, in which any quasi-projective scheme is represented, while maintaining its non-reduced structure. This yields a more subtle invariant, called the schemic Grothendieck ring, in which we can formulate a form of integration resembling Kontsevich's motivic integration via arc schemes. In view of its more functorial properties, we can present a characteristic-free proof of the rationality of the geometric Igusa zeta series for certain hypersurfaces, thus generalizing the ground-breaking work on motivic integration by Denef and Loeser. The construction uses first-order formulae, and some infinitary versions, called formularies.

1. INTRODUCTION

The classical Grothendieck ring $K_0(k)$ of an algebraically closed field k is defined as the quotient of the free Abelian group on varieties over k , modulo the relations $[X] - [X']$, if $X \cong X'$, and

$$(1) \quad [X] = [X - Y] + [Y],$$

if Y is a closed subvariety, for Y, X, X' varieties (=reduced, separated schemes of finite type over k). We will refer to the former relations as *isomorphism relations* and to the latter as *scissor relations*, in the sense that we “cut out Y from X .” In this way, we cannot just take the class of a variety, but of any constructible subset. Multiplication on $K_0(k)$ is then induced by the fiber product. In sum, the three main ingredients for building the Grothendieck ring are: an isomorphism relation, scissor relations, and a product. Only the former causes problems if one wants to generalize the construction of the Grothendieck ring to include not just classes of varieties, but also of finitely generated schemes (with their nilpotent structure). Put bluntly, we cannot cut a scheme in two, as there is no notion of a scheme-theoretic complement. To describe what this ought to be, we turn to model-theory.

To model-theorists, constructible subsets are nothing else than definable subsets (in view of quantifier elimination for algebraically closed fields). Moreover, union and intersection correspond to disjunction and conjunction of the corresponding formulae. Therefore, instead of working with the theory of algebraically closed fields, we could repeat the previous construction over any first-order theory T . However, now it is less obvious what it means for two formulae to be isomorphic. The most straightforward way is to introduce the notion of a definable isomorphism. However, even for the theory of algebraically closed fields, this yields a priori a different notion of isomorphism than the geometric one: whereas the former allows for arbitrary quantifier free formulae, the latter is given by polynomials, that is to say, of formulae of the form $y = f(x)$, which we will call *explicit formulae*. This

Date: October 5, 2009.

1991 *Mathematics Subject Classification.* 13D15;14C35;14G10;18F30.

observation suggests that we should consider not necessarily all first-order formulae, but also some restricted classes. This general construction is discussed in §3.

It is beneficial to develop the theory in a relative setup, so we work over an arbitrary affine, Noetherian scheme $X = \text{Spec } A$, instead of just over an algebraically closed field. To construct a generalized Grothendieck ring for schemes, a so-called *schemic Grothendieck ring*, we need to settle on a first-order theory \mathbf{T} . The classical Grothendieck ring is obtained by taking for \mathbf{T} the theory of algebraically closed fields that are also A -algebras. Alternatively, one could also have chosen the theory of all A -algebras without zero-divisors, and so, to include all schemes, we could simply replace this by the theory \mathbf{T}_A of all A -algebras. Refinements lead to more relations and hence more manageable Grothendieck rings, the most important one of which is the theory \mathbf{Art}_A of all Artinian local A -algebras (=algebras that have finite length as an A -module). Since we no longer have quantifier elimination, we also need to make a decision on which formulae we will allow, both for our definition of isomorphism as well as for the classes we want to study. Varieties, and more generally schemes, are given by equations, and so the family of formulae of the form $f_1 = \cdots = f_s = 0$, with $f_i \in A[x]$ will provide the proper candidate for our generalization to schemes; we call such formulae therefore *schemic*. We show that there is a one-one correspondence between schemic formulae in n free variables up to \mathbf{T}_A -equivalence,¹ and closed subschemes of \mathbb{A}_X^n (Theorem 4.1). In fact, this result remains true when working in the theory \mathbf{Art}_A . As for isomorphisms, we may take either the class of explicit isomorphisms, or the larger class of schemic isomorphisms, both choices leading to the same *schemic Grothendieck ring* $\mathbf{Gr}(X^{\text{sch}})$. There is an obvious ring homomorphism $\mathbf{Gr}(X^{\text{sch}}) \rightarrow K_0(X)$ to the classical Grothendieck ring $K_0(X)$ of X . The main result, Theorem 5.4, is that two affine schemes of finite type over X are isomorphic if and only if they have the same class in $\mathbf{Gr}(X^{\text{sch}})$.

However, if we want more relations to hold in our Grothendieck ring, we need to enlarge the family of formulae, and work in the appropriate theory. In §6, we explain how in order to define the class of a non-affine scheme, we need to work modulo the theory \mathbf{Art}_A in the larger class of *pp-formulae*, that is to say, existentially quantified schemic formulae. This is apparent already when dealing with basic open subsets: if $U = D(f)$ is the basic open subset of \mathbb{A}_X^n , that is to say, $U = \text{Spec}(A[x]_f)$, then as an abstract affine scheme, it is given by the schemic formula $f(x)z = 1$, where z is an extra variable, whereas as an open subset of \mathbb{A}_X^n , it is given by the pp-formula $(\exists z)f(x)z = 1$; the isomorphism between these two sets is only true modulo \mathbf{Art}_A , and is given by a (non-explicit) schemic formula. This leads to the *pp-Grothendieck ring* $\mathbf{Gr}(X^{\text{pp}})$ of X , where instead of quantifier free formulae, we take Boolean combinations of pp-formulae, up to schemic isomorphisms. To any scheme of finite type over X , we can, by taking an open affine covering, associate a unique element in $\mathbf{Gr}(X^{\text{pp}})$.

Unfortunately, the original scissor relation (1) is no longer valid. Indeed, the complement of an open $U \subseteq Y$ does not carry a unique closed scheme structure anymore. The solution is to take the limit over all these structures, yielding the formal completion \hat{Y}_Z , where Z is the underlying variety of $Y - U$. At the level of formulae (for simplicity, we assume $A = F$ is an algebraically closed field henceforth), the negation of the pp-formula defining a basic open subset is equivalent with an infinite disjunction of schemic formulae, having the property that in any Artinian F -algebra, the set defined by the disjunction is already definable by one of the disjuncts (but different models may require different

¹Two formulae are equivalent modulo a theory if they define the same subsets in each model of the theory.

disjuncts). Such an infinitary (whence non-first-order) disjunction will be termed a *formulary*. Replacing formulae by formularies in the definition of the pp-Grothendieck ring, yields the *infinitary Grothendieck ring* $\mathbf{Gr}^\infty(F^{\text{pp}})$, in which every formal scheme over F is represented by the class of some formulary, resurrecting the old scissor relation into a new one: for any closed immersion $Z \subseteq Y$ of schemes of finite type over X , we have $[Y] = [Y - Z] + [\widehat{Y}_Z]$. All this is explained in §8.

There is one more variant that will be considered here, called the *formal Grothendieck ring* $\mathbf{Gr}(F^{\text{form}})$, in which we revert to the reduced situation by factoring out the ideal generated by all $[\widehat{Y}_Z] - [Z]$ for all closed immersions $Z \subseteq Y$. However, we will only work in this latter quotient (in which any two schemes with the same underlying variety have the same class) after we have taken arcs (see below). This does make a difference, as can be seen already on easy examples (Table 1). The advantage is that we get back the original scissor relation (1), which makes it easier to invoke inductive arguments when proving rationality of the motivic Igusa zeta series (to be discussed below). The relation between all these Grothendieck rings is given by the following (ring) homomorphisms

$$(2) \quad \mathbf{Gr}(F^{\text{sch}}) \rightarrow \mathbf{Gr}(F^{\text{pp}}) \rightarrow \mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{form}}) \rightarrow \mathbf{Gr}(F^{\text{var}}).$$

To discuss our main application, the motivic rationality of the geometric Igusa zeta series, we introduce a weak version of motivic integration in §7. For any Artinian F -algebra R , and any affine scheme X over F , we define the *arc scheme* $\nabla_R X$ of X along $\text{Spec } R$ as the scheme whose S -rational points correspond to the $R \otimes_F S$ -rational points of X , for any Artinian F -algebra S . This generalizes the truncated arc space of a variety, which is obtained by taking $R = F[\xi]/\xi^n F[\xi]$ and ignoring the nilpotent structure. The *arc integral* $\int Z dX$ is then defined as the class $[\nabla_Z X]$ in $\mathbf{Gr}(F^{\text{pp}})$, and the main result is that it only depends only the classes of X and Z (unlike in the classical case). The *geometric Igusa zeta series of X along the germ (Y, P)* is then defined as the formal power series

$$\text{Igu}_X^{(Y, P)}(t) := \sum_n \left(\int J_P^n Y dX \right) t^n$$

in $\mathbf{Gr}(F^{\text{pp}})[[t]]$, where the n -th jet of a germ (Y, P) is defined as the Artinian scheme $J_P^n Y := \text{Spec}(\mathcal{O}_Y/\mathfrak{m}_P^n)$, with \mathfrak{m}_P the maximal ideal of the closed point P . For the remainder of this introduction, I will assume that (Y, P) is the germ of a point on a line, and simply write $\text{Igu}_X^{\text{geom}}$ for this zeta series. Under the homomorphism from (2), this power series becomes the Denef-Loeser geometric Igusa zeta series. The aim is to recover within the new framework their result that $\text{Igu}_X^{\text{geom}}$ is rational over the localization $\mathbf{Gr}(F^{\text{var}})_{\mathbb{L}}$, where \mathbb{L} is the class of an affine line, called the *Lefschetz class*. Their proof relies on Embedded Resolution of Singularities, and hence works in positive characteristic only for surfaces. In §§9 and 10, I will give examples of hypersurfaces, in any characteristic, for which we can derive the rationality of the Igusa zeta series (in fact, over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$), without any appeal to resolution of singularities. The proofs are, moreover, far more elementary and algorithmic in nature because of the functorial properties of our construction.

2. THE GROTHENDIECK GROUP OF A LATTICE

The most general setup in which one can define a Grothendieck group is the category of semi-lattices. Recall that a *lattice* Λ is a partially ordered set in which every finite subset has an infimum and a supremum. For any two elements $a, b \in \Lambda$, we let $a \wedge b$ and $a \vee b$ denote respectively the infimum and the supremum of $\{a, b\}$. If only infima exist, then we

call Λ a *semi-lattice*. Given a semi-lattice Λ , we call a finite subset $S \subseteq \Lambda$ *admissible* if it has a supremum a , in which case we call S a *covering* of a .

Scissor relations. For each $n > 1$, we define the *n-th scissor polynomial*

$$S_n(x) := 1 - \prod_{i=1}^n (1 - x_i) = x_1 + \cdots + x_n - \cdots + (-1)^{n-1} x_1 \cdots x_n \in \mathbb{Z}[x]$$

where $x = (x_1, \dots, x_n)$. Let \mathfrak{i}_n be the *generic idempotency ideal*, that is to say, the ideal in $\mathbb{Z}[x]$ generated by the relations $x_i^2 - x_i$, for $i = 1, \dots, n$. The following identities among scissor polynomials will be useful later:

2.1. Lemma. *For each n , we have an equivalence relation*

$$(3) \quad S_n(x_1, \dots, x_n) + S_{n-1}(x_1 x_n, \dots, x_{n-1} x_n) \equiv S_{n-1}(x_1, \dots, x_{n-1}) + x_n \pmod{\mathfrak{i}_n}.$$

in $\mathbb{Z}[x]$, with $x = (x_1, \dots, x_n)$. More generally, for $y = (y_1, \dots, y_m)$, we have

$$(4) \quad S_{n+m}(x_1, \dots, x_n, y_1, \dots, y_m) + S_{nm}(x_1 y_1, x_1 y_2, \dots, x_n y_m) \equiv S_n(x_1, \dots, x_n) + S_m(y_1, \dots, y_m) \pmod{\mathfrak{i}_{n+m}}.$$

Proof. Note that (3) is just a special case of (4). Let us first prove that in any ring D , we have an identity

$$(5) \quad \prod_{i=1}^n (1 - e a_i) = 1 - e + e \prod_{i=1}^n (1 - a_i)$$

for $a_i \in D$ and e an idempotent in D . Indeed, write $1 - e a_i = 1 - e + e(1 - a_i)$. Since $e(1 - e) = 0$, the expansion of the product on the left hand side of (5) yields $(1 - e)^n + e^n \prod_{i=1}^n (1 - a_i)$, and the claim follows since e and $1 - e$ are both idempotents.

To prove (4), we carry our calculations out in $C := \mathbb{Z}[x, y]/\mathfrak{i}_{n+m}$. To simplify notation, let us write $P := \prod_{i=1}^n (1 - x_i)$ and $Q := \prod_{j=1}^m (1 - y_j)$. Hence the first, third, and fourth term in (4) are respectively $1 - PQ$, $1 - P$ and $1 - Q$. Let us therefore expand the second term. Applying (5), for each i , with x_i as idempotent in the product indexed by j , and then again in the last line, with $1 - Q$ as idempotent, we get

$$\begin{aligned} 1 - S_{nm}(x_1 y_1, x_1 y_2, \dots, x_n y_m) &= \prod_{i=1}^n \left(\prod_{j=1}^m (1 - x_i y_j) \right) \\ &= \prod_{i=1}^n (1 - x_i + x_i Q) \\ &= \prod_{i=1}^n (1 - (1 - Q)x_i) = Q + (1 - Q)P \end{aligned}$$

From this, (4) now follows immediately. \square

We can write any scissor polynomial S_n as the difference $S_n^+ - S_n^-$ of two polynomials S_n^+ and S_n^- with positive coefficients, that is to say, S_n^+ is the sum of terms in S_n of odd degree, and S_n^- is minus the sum of all terms of even degree. Put differently, S_n^+ and S_n^- are the respective sums of all square-free monomials in the variables (x_1, \dots, x_n) of respectively odd and (positive) even degree.

The scissor group of a lattice. Given a lattice Λ , let $\mathbb{Z}[\Lambda]$ be the free Abelian group on Λ . Using the infimum of Λ as multiplication, we get a ring structure on $\mathbb{Z}[\Lambda]$, that is to say,

$$\left(\sum_{i=1}^n a_i\right) \cdot \left(\sum_{j=1}^m b_j\right) := \sum_{i=1}^n \sum_{j=1}^m a_i \wedge b_j$$

for $a_i, b_j \in \Lambda$. Let $\mathbf{a} := (a_1, \dots, a_n)$ be a tuple in Λ . We will write $\bigvee \mathbf{a}$ for $a_1 \vee \dots \vee a_n$. Substitution induces a ring homomorphism $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\Lambda]: x_i \mapsto a_i$. In particular, $S_n(\mathbf{a})$ is a well-defined element in $\mathbb{Z}[\Lambda]$, and we may abbreviate it as $S(\mathbf{a})$, since the arity is clear from the context. Note that, since any element of Λ is idempotent in $\mathbb{Z}[\Lambda]$, the kernel of this homomorphism $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\Lambda]$ contains \mathfrak{i} , the generic idempotency ideal. We define the *scissor relation* on \mathbf{a} as the formal sum

$$(6) \quad \left(\bigvee \mathbf{a}\right) - S(\mathbf{a}).$$

For instance, if $n = 2$, then the (second) scissor relation $(a_1 \vee a_2) - S_2(a_1, a_2)$ is equal to

$$(7) \quad (a_1 \vee a_2) - a_1 - a_2 + (a_1 \wedge a_2).$$

Similarly, for $n = 3$, we get

$$(a_1 \vee a_2 \vee a_3) - a_1 - a_2 - a_3 + (a_1 \wedge a_2) + (a_1 \wedge a_3) + (a_2 \wedge a_3) - (a_1 \wedge a_2 \wedge a_3).$$

We define the *scissor group* of Λ , denoted $\mathbf{Sciss}(\Lambda)$, as the quotient of $\mathbb{Z}[\Lambda]$ by the subgroup N generated by all second scissor relations (7). Although we later will make a notational distinction between an element and its class in the scissor group, at present, no such distinction is needed, and so we continue to write a for the image of $a \in \Lambda$ in $\mathbf{Sciss}(\Lambda)$.

2.2. Remark. The ring structure on $\mathbb{Z}[\Lambda]$, given by \wedge , descends to a ring structure on $\mathbf{Sciss}(\Lambda)$, since N is in fact an ideal. When we apply this to formulae in the next section, this ring structure on $\mathbf{Sciss}(\Lambda)$ will play a minor role, and instead, a different multiplication will be introduced.

2.3. Proposition. *For each tuple \mathbf{a} in Λ , we have a scissor relation*

$$\bigvee \mathbf{a} = S(\mathbf{a})$$

in $\mathbf{Sciss}(\Lambda)$.

Proof. We prove this by induction on the length $n \geq 2$ of $\mathbf{a} = (a_1, \dots, a_n)$, where the case $n = 2$ is just the definition. Since \mathfrak{i} lies in the kernel of $\mathbb{Z}[x] \rightarrow \mathbb{Z}[\Lambda]$, the equivalence (3) in Lemma 2.1 becomes an identity in the latter group, that is to say,

$$(8) \quad S_n(a_1, \dots, a_n) = S_{n-1}(a_1, \dots, a_{n-1}) - S_{n-1}(a_1 \wedge a_n, \dots, a_{n-1} \wedge a_n) + a_n.$$

Viewing $\bigvee \mathbf{a}$ as the disjunction $b \vee a_n$, where $b := a_1 \vee \dots \vee a_{n-1}$, the defining (second) scissor relation yields an identity

$$(9) \quad \bigvee \mathbf{a} = b + a_n - (b \wedge a_n) = (a_1 \vee \dots \vee a_{n-1}) + a_n - (a_1 \wedge a_n) \vee \dots \vee (a_{n-1} \wedge a_n).$$

Subtracting (8) from (9), the left hand side is $\bigvee \mathbf{a} - S_n(\mathbf{a})$, and the right hand side is equal to

$$\begin{aligned} & ((a_1 \vee \dots \vee a_{n-1}) - S_{n-1}(a_1, \dots, a_{n-1})) - \\ & ((a_1 \wedge a_n) \vee \dots \vee (a_{n-1} \wedge a_n) - S_{n-1}(a_1 \wedge a_n, \dots, a_{n-1} \wedge a_n)). \end{aligned}$$

By induction, both terms are zero in $\mathbf{Sciss}(\Lambda)$, and the result follows. \square

In particular, N is generated by all scissor relations, of any arity. We may generalize this to a semi-lattice Λ (with multiplication still given by \wedge) as follows. Let N be the subgroup of $\mathbb{Z}[\Lambda]$ generated by all expressions $a - S_n(\mathbf{b})$, where $\mathbf{b} = (b_1, \dots, b_n)$ (or rather its entries) ranges over all admissible coverings of a , that is to say, such that a is the supremum of the b_i . It is not hard to check that N is in fact an ideal, and the resulting residue ring is the scissor ring $\text{Sciss}(\Lambda) := \mathbb{Z}[\Lambda]/N$.

The Grothendieck group of a graph (semi-)lattice. Let Λ be a (semi-)lattice. By a (directed) graph on Λ we simply mean a binary relation E on Λ (we do not require it to be compatible with the join or the meet). We define the *Grothendieck group* of (Λ, E) , denoted $\mathbf{K}_E(\Lambda)$, as the factor group of $\text{Sciss}(\Lambda)$ modulo the subgroup generated by all elements of the form $a - b$ such that there is an edge from a to b . In other words, $\mathbf{K}_E(\Lambda)$ is the quotient of the free Abelian group $\mathbb{Z}[\Lambda]$ on Λ modulo the subgroup $N_{\text{sciss}} + N_E$, where $N_{\text{sciss}} = N$ is the group of scissor relations and N_E the group generated by all $a - b$ with $(a, b) \in E$.

If \tilde{E} is the equivalence relation generated by E (meaning that a is equivalent to b if there is an undirected path from a to b), then $N_E = N_{\tilde{E}}$, and hence both graphs have the same Grothendieck group, as do all intermediate graphs between E and \tilde{E} . Therefore, often, though not always, E will already be an equivalence relation. Although the quotient $\mathbb{Z}[\Lambda]/N_E$ is equal to the free Abelian group on the quotient Λ/\tilde{E} , the latter is no longer a semi-lattice, and so a priori, does not admit a scissor subgroup. We may paraphrase this situation as: *cut first, then identify*.

3. THE GROTHENDIECK RING OF A THEORY

Inspired by the ground-breaking work of Denef and Loeser, model-theorists have recently been interested in the Grothendieck ring of an arbitrary first-order theory, see for instance [2, 3, 10, 11]. The new perspective offered here is that rather than looking at all formulae and all definable isomorphisms, much better behaved objects can be obtained when restricting these classes.

Fix a language \mathcal{L} , by which we mean the collection of all well-formed formulae in a certain signature, in a fixed countable collection of variables v_1, v_2, \dots .² Note that some authors use the terms language and signature interchangeably. We denote a formula by Greek lower case letters ϕ, ψ, \dots , and often we give names to their free variables as well, taken from the last letters of the Latin alphabet: x, y, z . If $\phi(x_1, \dots, x_s)$ is an \mathcal{L} -formula, and M an \mathcal{L} -structure, then the *set defined by ϕ in M* , or the *interpretation of ϕ in M* is the following subset $\phi(M)$. Suppose $x_i = v_{n_i}$, and let n be the maximum of all n_i . Then $\phi(M)$ is the subset of M^n of all (a_1, \dots, a_n) such that $\phi(a_{n_1}, \dots, a_{n_s})$ holds in M . Any set of the form $\phi(M)$ will be called a *definable subset*. Note that the Cartesian power n is not determined by the number of free variables s , but by the highest index of a v_i occurring in the formula ϕ ; we call n the *arity* of ϕ (which therefore is not to be confused with its number of free variables). For instance, the subset defined by $\phi := (v_3 = 0) \wedge (v_7 = 1)$ in \mathbb{Z} is the 7-ary subset $\mathbb{Z}^2 \times \{0\} \times \mathbb{Z}^3 \times \{1\}$ of \mathbb{Z}^7 . Also note that this leaves a certain amount of ambiguity: the formula $v_3 = 0$ has, prima facie, arity 3, but as a conjunct of ϕ it behaves as a formula of arity 7. Notwithstanding all this, the tacit rule will be that if (x_1, \dots, x_n) denotes the tuple of free variables of ϕ , then (x_1, \dots, x_n) stands for the tuple (v_1, \dots, v_n) , or more generally, if x, y, z, \dots are tuples of free variables of ϕ , which are listed in that order, and whose total number equals n , then these variables represent the first n variables

²We usually start numbering from 1, as any non-logician would.

v_i , that is to say, $(x_1, \dots, x_s, y_1, \dots, y_t, z, \dots) = (v_1, \dots, v_n)$. Put differently, unless mentioned explicitly, the arity of a formula is its number of free variables. In this respect, it is useful to introduce the *primary form* of ϕ , defined as $\phi^\circ := \phi(v_1, \dots, v_s)$, where s is the number of free variables of ϕ . The implicit assumption is that a formula is in primary form, unless the variables are stated explicitly. Furthermore, the implicit assumption is that the disjunction or conjunction of two formulae is equal to the maximum of their arities.

This definition also applies to a sentence σ , that is to say, a formula without free variables: given an \mathcal{L} -structure M , we let $\sigma(M)$ be one of the two possible subsets of $M^0 := \{\emptyset\}$, namely $\{\emptyset\}$ if σ holds in M , and \emptyset if it does not. If ϕ and ψ are two \mathcal{L} -formulae, then $\phi \wedge \psi$ and $\phi \vee \psi$ are again \mathcal{L} -formulae, defining in each \mathcal{L} -structure M respectively $\phi(M) \cap \psi(M)$ and $\phi(M) \cup \psi(M)$. Moreover, $\neg\phi$ defines the complement of $\phi(M)$ in M^n , where n is the arity of ϕ . A trivial yet important formula is the *n-th Lefschetz formula*

$$\lambda_n := (v_1 = v_1) \wedge \dots \wedge (v_n = v_n),$$

defining the full Cartesian power M^n in any model. We abbreviate the Lefschetz formula $\lambda_n(x) := (x = x)$ with x an n -tuple of variables (which, according to our tacit assumptions, stand for the n first variables v_i), and write $\lambda(x)$ for $\lambda_1(x)$.

An \mathcal{L} -theory \mathbf{T} is any non-empty collection of consistent \mathcal{L} -sentences (it is convenient to assume that \mathbf{T} contains at least one sentence, which we always could assume to be a tautology like $\forall x \lambda(x)$). A *model* of \mathbf{T} is an \mathcal{L} -structure for which all sentences in \mathbf{T} are true. By the compactness theorem, every theory has at least one model. Given a (non-empty) collection \mathfrak{K} of \mathcal{L} -structures, we define the \mathcal{L} -theory of \mathfrak{K} , denoted $\mathbf{T}_{\mathcal{L}}(\mathfrak{K})$, to be the collection of all \mathcal{L} -sentences that are true in any structure belonging to \mathfrak{K} . The collection \mathfrak{K} is *axiomatizable* (also called *first-order*), if it consists precisely of the models of $\mathbf{T}_{\mathcal{L}}(\mathfrak{K})$.

Let \mathbf{T} be a theory in the language \mathcal{L} . We say that two formulae ϕ and ψ are *T-equivalent*, denoted $\phi \sim_{\mathbf{T}} \psi$, if they define the same subset in any model of \mathbf{T} , that is to say, if $\phi(M) = \psi(M)$, for any model M of \mathbf{T} . By the compactness theorem, this is equivalent with \mathbf{T} proving that $(\forall x)\phi(x) \leftrightarrow \psi(x)$. In particular, equivalent formula must have the same free variables v_i . Note that the logical connectives \wedge and \vee , as well as negation \neg , respect this equivalence relation. If \mathbf{T} consists entirely of tautologies, then two formulae are \mathbf{T} -equivalent if and only if they are logically equivalent. If $\mathbf{T} = \mathbf{T}_{\mathcal{L}}(\mathfrak{K})$ is the theory of a non-empty class \mathfrak{K} of \mathcal{L} -language, then ϕ and ψ are \mathbf{T} -equivalent if and only if they define the same subset in each structure belonging to \mathfrak{K} . In other words, we do not need to check all models, but only those that “generate” the theory, and therefore, we will often make no distinction between theories and collections of structures. (Caveat: when dealing with infinitary formulae, as in §8, this is no longer true.) For instance, instead of calling two formulae $\mathbf{T}_{\mathcal{L}}(\mathfrak{K})$ -equivalent, we may just simply call them \mathfrak{K} -equivalent. When the theory \mathbf{T} is fixed, we will often identify \mathbf{T} -equivalent formulae. Formally, we therefore introduce:

The category of definable sets. The *category of T-definable sets*, $\text{Def}(\mathbf{T})$, has as objects the \mathbf{T} -equivalence classes of formulae and as morphisms the definable maps. There, are in fact a few variant ways of defining the latter notion, and for our purposes, the following will be most suitable. Let ϕ and ψ be \mathcal{L} -formulae of arity n and m respectively, and let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be variables. A formula $\theta(x, y)$ (or, rather, its \mathbf{T} -equivalence class) is called *morphic* (on ϕ), or *defines a T-morphism* $f_\theta: \phi \rightarrow \psi$, if \mathbf{T} proves the following sentences

$$(1) (\forall x)(\exists y)\phi(x) \Rightarrow \theta(x, y)$$

- (2) $(\forall x, y, y') \phi(x) \wedge \theta(x, y) \wedge \theta(x, y') \Rightarrow y = y'$
 (3) $(\forall x, y) \phi(x) \wedge \theta(x, y) \Rightarrow \psi(x)$.

In other words, (1) and (2) express that in each model M of \mathbf{T} , the definable subset $\theta(M) \subseteq M^n \times M^m$, when restricted to $\phi(M) \times M^m$, is the graph of a function $f_M: \phi(M) \rightarrow M^m$, and, furthermore, (3) ensures that the image of f_M lies in $\psi(M) \subseteq M^m$. We will therefore denote this map by $f_M: \phi(M) \rightarrow \psi(M)$. Although slightly inaccurate, we will express this situation also by simply saying that $\theta(M)$ is the graph of a function $\phi(M) \rightarrow \psi(M)$. As part of the definition of morphic formula is an, often implicit, division of the free variables in two sets, the *source* variables, x , and the *target* variables, y . The next result shows that these definitions constitute a category.

3.1. Lemma. *If $f_\theta: \phi(x) \rightarrow \psi(y)$ and $g_\zeta: \psi(y) \rightarrow \gamma(z)$ are morphisms defined by $\theta(x, y)$ and $\zeta(y, z)$ respectively, then the formula $(\zeta \circ \theta)(x, z) := (\exists y) \theta(x, y) \wedge \zeta(y, z)$ is again a morphic formula, defining the composition $h: \phi(x) \rightarrow \gamma(z)$.*

Proof. The proof is straightforward but technical; since we will rely on it heavily, we will provide it in some detail. Put $\xi := \zeta \circ \theta$. We first verify condition (1), and to this end, we may work in a fixed model M of \mathbf{T} . So, let $\mathbf{a} \in \phi(M)$. By (1) and (3) applied to $\theta(x, y)$, we get a tuple $\mathbf{b} \in \psi(M)$ such that $(\mathbf{a}, \mathbf{b}) \in \theta(M)$. By the same argument, applied to ζ , we then get $\mathbf{c} \in \gamma(M)$ such that $(\mathbf{b}, \mathbf{c}) \in \zeta(M)$. In particular, \mathbf{b} witnesses that $(\mathbf{a}, \mathbf{c}) \in \xi(M)$, proving (1) for ξ . The same argument essentially also shows that (3) holds. So remains to show (2) for ξ . To this end, let $\mathbf{a} \in \phi(M)$ such that (\mathbf{a}, \mathbf{c}) and $(\mathbf{a}, \mathbf{c}')$ both belong to $\xi(M)$. This means that there are tuples \mathbf{b}, \mathbf{b}' such that $\theta(\mathbf{a}, \mathbf{b}) \wedge \zeta(\mathbf{b}, \mathbf{c})$ and $\theta(\mathbf{a}, \mathbf{b}') \wedge \zeta(\mathbf{b}', \mathbf{c}')$ hold in M . By condition (2) for θ , we get $\mathbf{b} = \mathbf{b}'$, and by (3), this tuple then belongs to $\psi(M)$. So we may repeat this argument to the tuples $\mathbf{b}, \mathbf{c}, \mathbf{c}'$ and ζ , to get $\mathbf{c} = \mathbf{c}'$, as we wanted to show. \square

We call a morphism $f_\theta: \phi \rightarrow \psi$, or the corresponding formula θ , *injective*, *surjective*, or *bijective*, if all the corresponding maps $f_M: \phi(M) \rightarrow \psi(M)$ are. The next result shows that any definable bijection is an isomorphism in the category $\text{Def}(\mathbf{T})$, that is to say, its inverse is also a morphism, which we therefore call a **T-isomorphism**.

3.2. Lemma. *Let $\theta(x, y)$ be a morphic formula defining a bijection $f_\theta: \phi(x) \rightarrow \psi(y)$. If $\text{inv}(\theta)(x, y) := \theta(y, x)$, then $\text{inv}(\theta)$ defines a morphism $g: \psi \rightarrow \phi$, which gives the inverse of f_M on each model M of \mathbf{T} .*

Proof. To show that $\zeta(x, y) := \text{inv}(\theta)(x, y)$ is morphic, yielding a morphism $\psi \rightarrow \phi$, we may again check this in a model M of \mathbf{T} . Suppose $\mathbf{a} \in \psi(M)$. Since f_M is bijective, there is some $\mathbf{b} \in \phi(M)$ such that $f_M(\mathbf{b}) = \mathbf{a}$. This means that $(\mathbf{b}, \mathbf{a}) \in \theta(M)$, whence $(\mathbf{a}, \mathbf{b}) \in \zeta(M)$, proving conditions (1) and (3). Since f_M is a bijection, the tuple \mathbf{b} is unique, and this proves (2). It is now easy to see that the map $g_M: \mathbf{a} \mapsto \mathbf{b}$ is the inverse of f_M . \square

3.2.1. \mathcal{I} -morphisms. Given a family $\mathcal{I} \subseteq \mathcal{L}$ of formulae (closed under \mathbf{T} -equivalence), by an \mathcal{I} -definable **T-map**, or simply, an \mathcal{I} -morphism between ϕ and ψ , we mean a morphic formula θ belonging to \mathcal{I} which defines a morphism $f_\theta: \phi \rightarrow \psi$ (without imposing any restriction on ϕ and ψ). We call $f_\theta: \phi \rightarrow \psi$ an \mathcal{I} -isomorphism (modulo \mathbf{T}), if f_θ is bijective and its inverse is again an \mathcal{I} -morphism. In view of Lemma 3.2, a bijective morphism f_θ is an \mathcal{I} -isomorphism, if, for instance, both θ and $\text{inv}(\theta)$ belong to \mathcal{I} . However, in general, a bijective \mathcal{I} -morphism need not be an \mathcal{I} -isomorphism (see, for instance, Example 4.7).

A note of caution: in general, \mathcal{I} -isomorphism, in spite of its name, is not an equivalence relation, since it is not clear that the composition of two \mathcal{I} -isomorphism is again an

\mathcal{I} -morphism. In view of Lemma 3.1, the collection of \mathcal{I} -morphisms is closed under composition, if, for instance, \mathcal{I} is closed under existential quantification, although, as we shall see, this is not the only instance in which this is true.

3.2.2. Explicit formulae. A morphic formula, in general, only defines a partial map, but there is an important type of morphism that is always global: by an *explicit* morphism, we mean a formula $\theta(x, y)$ of the form $\bigwedge_{j=1}^m y_j = t_j(x)$, with each t_j an \mathcal{L} -term, and with source variables $x = (x_1, \dots, x_n)$ and target variables $y = (y_1, \dots, y_m)$; we will abbreviate this by $y = t(x)$, for $t := (t_1, \dots, t_m)$. Such a formula always defines a global map on each model M , given as $t_M: M^n \rightarrow M^m: \mathbf{a} \mapsto t(\mathbf{a})$. In particular, if $f: \phi \rightarrow \psi$ is explicit, then f_M is the restriction of t_M to $\phi(M)$. We denote the collection of all explicit formulae by \mathcal{Expl} . Note that \mathcal{Expl} is compositionally closed, for if $y = t(x)$ and $z = s(y)$ are two explicit formulae, then the explicit formula $z = s(t(x))$ defines their composition. However, as Example 4.7 shows, a bijective \mathcal{Expl} -morphism need not be an \mathcal{Expl} -isomorphism. At any rate, any formula ϕ is \mathcal{Expl} -isomorphic to its primary form ϕ° .

We already observed that the logical connectives \wedge and \vee are well-defined on $\text{Def}(\mathbf{T})$. We introduce two further operations on $\text{Def}(\mathbf{T})$.

3.2.3. Multiplication on $\text{Def}(\mathbf{T})$. Let ϕ and ψ be two \mathcal{L} -formulae, in n and m free variables respectively (or, more correctly, of arity n and m respectively). We define their *product* as

$$(\phi \times \psi)(v_1, \dots, v_{n+m}) := \phi(v_1, \dots, v_n) \wedge \psi(v_{n+1}, \dots, v_{n+m}).$$

A note of caution: it is not always true that $(\phi \times \psi)(M)$ is equal to $\phi(M) \times \psi(M)$, in a model M of \mathbf{T} , due to the numbering of the variables. This only holds for primary forms since always $\phi \times \psi = \phi^\circ \times \psi^\circ$, and the interpretation of this formula in M is equal to $\phi^\circ(M) \times \psi^\circ(M)$. In particular, although this multiplication is not Abelian, it is up to explicit isomorphism (given by permuting the variables appropriately). For Lefschetz formulae we obviously have $\lambda_n \times \lambda_m = \lambda_{m+n}$.

We leave it to the reader to verify that if ϕ_i and ψ_i are \mathbf{T} -equivalent, for $i = 1, 2$, then so are the respective products $\phi_1 \times \phi_2$ and $\psi_1 \times \psi_2$. In other words, the multiplication is well-defined modulo \mathbf{T} -equivalence, and hence yields a multiplication on $\text{Def}(\mathbf{T})$. The multiplicative unit in $\text{Def}(\mathbf{T})$ is the class of any sentence σ which is a logical consequence of \mathbf{T} , and will be denoted \top (for instance, one may take σ to be the tautology $(\forall x)\lambda(x)$). We will also write \perp for the class of $\neg\sigma$, and we have $\perp \times \phi = \perp = \phi \times \perp$, for all formulae ϕ (note that, per convention, the Cartesian product of any set with the empty set is the empty set).

Given two morphisms $f_\theta: \phi \rightarrow \psi$ and $f_{\theta'}: \phi' \rightarrow \psi'$ of \mathcal{L} -formulae, they induce a morphism $f: (\phi \times \phi') \rightarrow (\psi \times \psi')$ between the respective products as follows. If $\theta(\mathbf{x}, \mathbf{y})$ and $\theta'(\mathbf{x}', \mathbf{y}')$ are the respective defining formulae, then the order of the variables in the product $\theta \times \theta'$ is by definition $\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}'$. The formula obtained from this product by changing this order to $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}'$ then defines f . Note that the formula defining f is therefore \mathcal{Expl} -isomorphic with $\theta \times \theta'$: indeed, this isomorphism is given by

$$(10) \quad \delta(\mathbf{x}, \mathbf{y}, \mathbf{x}', \mathbf{y}, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4) := (\mathbf{z}_1 = \mathbf{x}) \wedge (\mathbf{z}_2 = \mathbf{x}') \wedge (\mathbf{z}_3 = \mathbf{y}) \wedge (\mathbf{z}_4 = \mathbf{y}')$$

with $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ tuples of variables (of the appropriate length).

3.2.4. Disjoint sum. To define the second operation on $\text{Def}(\mathbf{T})$, we need to assume that \mathcal{L} contains at least two constant symbols which are interpreted in each model of \mathbf{T} as different elements. Since in all our applications, \mathbf{T} will always be a theory of rings, for

which we have the distinct constants 0 and 1, we will for simplicity assume that these two constant symbols are denoted 0 and 1 (not to be confused with the \mathbf{T} -equivalence classes \perp and \top). We define the *disjoint sum* of two formulae ϕ and ψ as the formula

$$\phi \oplus \psi := (\phi \wedge (v_{n+1} = 0)) \vee (\psi \wedge (v_{n+1} = 1)),$$

where n is the maximum of the arities of ϕ and ψ (so that v_{n+1} is the “next” free variable). The disjoint sum is a commutative operation, but without identity element: we only have that $\phi \oplus \perp$ is $\mathcal{E}xpl$ -isomorphic with ϕ via the morphic formula $y = x$ (note that its inverse $\phi \rightarrow \phi \oplus \perp$ is given by $(y = x) \wedge (v_{n+1} = 0)$). Unlike \vee , disjoint sum is not idempotent: $\phi \oplus \phi$ will in general be different from ϕ . We will assume that the (set-theoretic) *disjoint union* $V \sqcup W$ of two sets V and W is defined as the union of $V \times \{0\}$ and $W \times \{1\}$, so that we proved the following characterization of disjoint sum:

3.3. Lemma. *Given two formulae ϕ and ψ , their direct sum $\phi \oplus \psi$ is, up to \mathbf{T} -equivalence, the unique formula γ such that*

$$\gamma(M) = \phi(M) \sqcup \psi(M),$$

for all models M of \mathbf{T} . □

The Grothendieck ring of a theory. It is useful to construct Grothendieck rings over restricted classes of formulas, like quantifier free, or pp-formulae. To do this in as general a setup as possible, fix a language \mathcal{L} with constant symbols for 0 and 1 (so that disjoint sums are defined). By a *sub-semi-lattice* \mathcal{G} of $\text{Def}(\mathbf{T})$, we mean a collection of formulae closed under conjunction. If \mathcal{G} is also closed under disjunction, we call it a *sublattice*, and if it is moreover closed under negation, we call it *Boolean*.³ We say that a sub-semi-lattice \mathcal{G} is *primary* if it contains the Lefschetz formula λ and all formulae of the form $v_i = c$, with c a constant, and, moreover, ϕ belongs to \mathcal{G} if and only if its primary form ϕ° does. It follows that \mathcal{G} is closed under multiplication, and if \mathcal{G} is a lattice, then it is also closed under disjoint sums.

For the remainder of this section, we fix two primary sub-semi-lattices $\mathcal{F} \subseteq \text{Def}(\mathbf{T})$ (the “formulae”) and $\mathcal{I} \subseteq \text{Def}(\mathbf{T})$ (the “isomorphisms”). We assume, moreover, that $\mathcal{E}xpl \subseteq \mathcal{I}$; and, more often than not, \mathcal{F} will actually be a lattice. In any case, by §2, we can define the scissor group $\text{Sciss}(\mathcal{F})$. The \mathcal{I} -isomorphism relation induces a binary relation on \mathcal{F} , which we denote by $\cong_{\mathcal{I}}$ (strictly speaking, we should consider the equivalence relation generated by this relation, but this does not matter when working with Grothendieck groups). Recall that the corresponding Grothendieck group $\mathbf{K}_{\cong_{\mathcal{I}}}(\mathcal{F})$ is the quotient of $\mathbb{Z}[\mathcal{F}]$ modulo the subgroup $\mathfrak{N} := \mathfrak{N}_{\mathcal{I}} + \mathfrak{N}_{\text{sciss}}$, where $\mathfrak{N}_{\mathcal{I}}$ is generated by all expressions $\langle \phi \rangle - \langle \phi' \rangle$, for any pair of \mathcal{I} -isomorphic formulae $\phi, \phi' \in \mathcal{F}$, and $\mathfrak{N}_{\text{sciss}}$ is generated, in the lattice case, by all second scissor relations $\langle \phi \vee \psi \rangle - \langle \phi \rangle - \langle \psi \rangle + \langle \phi \wedge \psi \rangle$, for any pair $\phi, \psi \in \mathcal{F}$, and where for clarity, we have written $\langle \phi \rangle$ for the \mathbf{T} -equivalence class of ϕ (although we will continue our practice of confusing \mathbf{T} -equivalence classes with their representatives). If \mathcal{F} is only a semi-lattice, then $\mathfrak{N}_{\text{sciss}}$ is generated by all scissor relations $\langle \psi \rangle - \langle S_n(\phi_1, \dots, \phi_n) \rangle$, for all $\phi_i \in \mathcal{F}$ such that $\langle \psi \rangle = \langle \phi_1 \vee \dots \vee \phi_n \rangle$.

3.4. Lemma. *The subgroup \mathfrak{N} is a two-sided ideal in $\mathbb{Z}[\mathcal{F}]$ with respect to the multiplication \times on formulae, and the quotient ring is commutative.*

Proof. Note that in §2 we used the multiplication given by \wedge to define scissor relations, whereas here multiplication is as defined in §3.2.3, and is not commutative. Let α, ϕ, ψ be formulae in \mathcal{F} . If $\phi \cong_{\mathcal{I}} \psi$, then $\phi \times \alpha \cong_{\mathcal{I}} \psi \times \alpha$ by the discussion of (10), showing that

³Although only the restrictions $\mathcal{G} \cap \mathcal{L}_n$ are then Boolean lattices, this should not cause any confusion.

$\langle \alpha \rangle \times (\langle \phi \rangle - \langle \psi \rangle) \in \mathfrak{N}_{\mathcal{I}}$, and a similar result for multiplication from the right. Hence $\mathfrak{N}_{\mathcal{I}}$ is a two-sided ideal. Moreover, since $\alpha \times \phi$ is explicitly isomorphic, whence \mathcal{I} -isomorphic, to $\phi \times \alpha$, both formulae have the same image modulo $\mathfrak{N}_{\mathcal{I}}$, showing that multiplication is commutative in the quotient.

For simplicity, we only give the proof that \mathfrak{N} is an ideal in the lattice case, and leave the semi-lattice case to the reader. It suffices to show that any multiple of

$$u := \langle \phi \vee \psi \rangle - \langle \phi \rangle - \langle \psi \rangle + \langle \phi \wedge \psi \rangle$$

lies in \mathfrak{N} . Choose α' to be \mathcal{I} -isomorphic to α , but having free variables distinct from those of ϕ and ψ (this can always be accomplished with an explicit change of variables). It follows from what we just proved that $\langle \alpha \rangle \times u - \langle \alpha' \rangle \times u$ belongs to \mathfrak{N} , and so we may assume from the start that α has no free variables in common with ϕ and ψ . In particular, $\langle \alpha \rangle \times \langle \phi \rangle = \langle \phi \wedge \alpha \rangle$, and similarly for any other term in u . It follows that

$$(11) \quad \langle \alpha \rangle \times u = \langle (\phi \wedge \alpha) \vee (\psi \wedge \alpha) \rangle - \langle \phi \wedge \alpha \rangle - \langle \psi \wedge \alpha \rangle + \langle (\phi \wedge \alpha) \wedge (\psi \wedge \alpha) \rangle$$

which is none other than the second scissor relation on the two formulae $\phi \wedge \alpha$ and $\psi \wedge \alpha$, and therefore, by definition, lies in \mathfrak{N} . \square

Note that $\mathfrak{N}_{\text{sciss}}$ is in general not an ideal. In any case, $\mathbf{K}_{\cong_{\mathcal{I}}}(\mathcal{F})$ has the structure of a commutative ring, which we will call the \mathcal{I} -Grothendieck ring of \mathcal{F} -formulae modulo \mathbf{T} and which we denote, for simplicity, by $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$, or just $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$, if the underlying theory \mathbf{T} is understood. We denote the class of a formula ϕ in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ by $[\phi]$, or in case we want to emphasize the isomorphism type, by $[\phi]_{\mathcal{I}}$. We denote the class of \perp and \top by 0 and 1 respectively; they are the neutral elements for addition and multiplication in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ respectively. Note that 1 is equal to the class of $v_1 = 0$ defining a singleton.

The full Grothendieck ring of a theory \mathbf{T} is obtained by taking for \mathcal{F} and \mathcal{I} simply all formulae. It will be denoted $\mathbf{Gr}(\mathbf{T})$. Immediately from the definitions, we have:

3.5. Corollary. *For each pair $\mathcal{F}, \mathcal{I} \subseteq \text{Def}(\mathbf{T})$ as above, there exists a canonical additive epimorphism of groups $\text{Sciss}(\mathcal{F}) \rightarrow \mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$.*

Moreover, if \mathbf{T}' is a subtheory of \mathbf{T} , and $\mathcal{E}xpl \subseteq \mathcal{I}' \subseteq \mathcal{I}$ and $\mathcal{F}' \subseteq \mathcal{F}$ primary sub-semi-lattices, then we have a natural homomorphism of Grothendieck rings $\mathbf{Gr}_{\mathcal{I}'}^{\mathbf{T}'}(\mathcal{F}') \rightarrow \mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$. \square

Note that even if $\mathbf{T} = \mathbf{T}'$ and $\mathcal{I} = \mathcal{I}'$, the latter homomorphism need not be injective, as there are potentially more relations when the class of formulae is larger.

The Lefschetz class. We denote the class of the first Lefschetz formula $\lambda := (v_1 = v_1)$ by \mathbb{L} (recall that by assumption λ belongs to \mathcal{F}), and call it the *Lefschetz class* of $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$. By definition of product, we immediately get:

3.6. Lemma. *For every n , we have $[\lambda_n] = \mathbb{L}^n$ in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$.* \square

3.7. Lemma. *If \mathcal{F} is a lattice, then $[\phi \oplus \psi] = [\phi] + [\psi]$ in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$, for all $\phi, \psi \in \mathcal{F}$. If \mathcal{F} is moreover Boolean, then $[\neg \phi] = \mathbb{L}^n - [\phi]$, where n is the arity of ϕ .*

Proof. To prove the first assertion, let $\phi'(x, z) := \phi(x) \wedge (z = 0)$ and $\psi'(x, z) := \psi(x) \wedge (z = 1)$. By definition, and after possibly taking primary forms, we have $[\phi' \vee \psi'] = [\phi \oplus \psi]$. The claim now follows since $\phi' \wedge \psi'$ defines the empty set in any model M of \mathbf{T} , and hence its class is zero. To prove the second, observe that $\langle \phi \wedge \neg \phi \rangle = 0$, whereas $\phi \vee \neg \phi$ is \mathbf{T} -equivalent with λ_n . Hence the result follows by Lemma 3.6. \square

3.8. Lemma. *If \mathcal{F} is Boolean, then the image of the ideal $\mathfrak{N}_{\text{sciss}}$ in $Z := \mathbb{Z}[\mathcal{F}]/\mathfrak{N}_{\mathcal{I}}$ is generated as a group by all formal sums of the form $\langle \phi \rangle + \langle \psi \rangle - \langle \phi \oplus \psi \rangle$, with ϕ and ψ formulae in \mathcal{F} .*

Proof. Let \mathfrak{N}' be the set of all formal sums in Z of scissor relations $\langle \alpha \rangle + \langle \beta \rangle - \langle \alpha \oplus \beta \rangle$, with α and β in \mathcal{F} . Let ϕ and ψ be two formulae in \mathcal{F} . Then the scissor relation $\langle \phi \rangle + \langle \psi \rangle - \langle \phi \wedge \psi \rangle - \langle \phi \vee \psi \rangle$ is equal to the difference

$$\left(\langle \phi \rangle + \langle \neg \phi \wedge \psi \rangle - \langle \phi \vee \psi \rangle \right) - \left(\langle \neg \phi \wedge \psi \rangle + \langle \phi \wedge \psi \rangle - \langle \psi \rangle \right)$$

whence belongs to \mathfrak{N}' , since $\phi \vee \psi$ is \mathcal{I} -isomorphic with $\phi \oplus (\neg \phi \wedge \psi)$. Using (11), it is clear that \mathfrak{N}' is closed under multiples, whence is an ideal of Z . \square

In particular, \mathfrak{N} is generated as a group in $\mathbb{Z}[\mathcal{F}]$ by $\mathfrak{N}_{\mathcal{I}}$ and all formal sums $\langle \phi \rangle + \langle \psi \rangle - \langle \phi \oplus \psi \rangle$, for $\phi, \psi \in \mathcal{F}$.

3.9. Corollary. *Suppose \mathcal{F} is a Boolean lattice. If $\mathcal{G} \subseteq \mathcal{F}$ is a sub-semi-lattice whose Boolean closure is equal to \mathcal{F} , then the natural map $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{G}) \rightarrow \mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ is surjective, and hence $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ is generated as a group by classes of formulae in \mathcal{G} . If \mathcal{G} is moreover closed under disjoint sums, then any element in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ is of the form $[\psi] - [\psi']$, with $\psi, \psi' \in \mathcal{G}$.*

Proof. Let G be the image of $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{G}) \rightarrow \mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ (see Corollary 3.5), that is to say, the subgroup generated by all $[\phi]$ with $\phi \in \mathcal{G}$. By Lemma 3.7, the class of the negation of a formula in \mathcal{G} lies in G . Since every term in $[S_n(\phi_1, \dots, \phi_n)]$, for $\phi_i \in \mathcal{G}$, lies by assumption in G , so does the class of any disjunction $\phi_1 \vee \dots \vee \phi_n$ by Proposition 2.3 and Corollary 3.5. This proves the first assertion.

To prove the last, we can write, by what we just proved, any element u of $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ as a sum $[\phi_1] + \dots + [\phi_s] - [\phi_{s+1}] - \dots - [\phi_t]$, with $\phi_i \in \mathcal{G}$. Let ψ be the disjoint sum $\phi_1 \oplus \dots \oplus \phi_s$ and let ψ' be the disjoint sum $\phi_{s+1} \oplus \dots \oplus \phi_t$. By assumption, both ψ and ψ' lie in \mathcal{G} , and $u = [\psi] - [\psi']$ by Lemma 3.8. \square

We call two \mathcal{F} -formulae ϕ_1 and ϕ_2 *stably \mathcal{I} -isomorphic in \mathcal{F} (modulo \mathbf{T})*, if there exists an \mathcal{F} -formula ψ such that $\phi_1 \oplus \psi \cong_{\mathcal{I}} \phi_2 \oplus \psi$. A priori, this is weaker than being \mathcal{I} -isomorphic, but in many cases, as we shall see, it is equivalent to it. In any case, we have:

3.10. Lemma. *Suppose \mathcal{F} is Boolean. If $\phi_1 \oplus \psi \cong_{\mathcal{I}} \phi_2 \oplus \psi$ and $\psi \Rightarrow \psi'$ modulo \mathbf{T} , then $\phi_1 \oplus \psi' \cong_{\mathcal{I}} \phi_2 \oplus \psi'$.*

Proof. By adding a disjoint copy to either side, we get

$$\phi_1 \oplus \psi \oplus (\psi' \wedge \neg \psi) \cong_{\mathcal{I}} \phi_2 \oplus \psi \oplus (\psi' \wedge \neg \psi).$$

Since $\psi \Rightarrow \psi'$, the formulae ψ' and $\psi \oplus (\psi' \wedge \neg \psi)$ are \mathcal{I} -isomorphic. \square

3.11. Theorem. *Suppose \mathcal{F} is Boolean, and $\cong_{\mathcal{I}}$ is an equivalence relation. Two formulae $\phi, \psi \in \mathcal{F}$ have the same class in $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F})$ if and only if they are stably \mathcal{I} -isomorphic in \mathcal{F} .*

Proof. One direction is easy, so that we only need to verify the direct implication. Let $\bar{\mathcal{F}}$ be the quotient of \mathcal{F} modulo the equivalence $\cong_{\mathcal{I}}$, that is to say, the collection of \mathcal{I} -isomorphism classes of formulae in \mathcal{F} . Let $Z := \mathbb{Z}[\mathcal{F}]/\mathfrak{N}_{\mathcal{I}}$, so that $\mathbf{Gr}_{\mathcal{I}}^{\mathbf{T}}(\mathcal{F}) \cong Z/\mathfrak{N}_{\text{sciss}}Z$. Moreover, the quotient map $\mathcal{F} \rightarrow \bar{\mathcal{F}}$ induces an isomorphism $Z \cong \mathbb{Z}[\bar{\mathcal{F}}]$, so that as an

Abelian group, Z is freely generated. By Lemma 3.8, we can find \mathcal{F} -formulae ϕ_i, ϕ'_i, ψ_i , and ψ'_i (without loss of generality we may assume that their number is the same), such that

$$(12) \quad \langle \phi \rangle + \sum_i \langle \phi_i \rangle + \langle \phi'_i \rangle + \langle \psi_i \oplus \psi'_i \rangle = \langle \psi \rangle + \sum_i \langle \psi_i \rangle + \langle \psi'_i \rangle + \langle \phi_i \oplus \phi'_i \rangle$$

in Z . Let σ be the disjoint sum of all the formulae ϕ_i, ϕ'_i, ψ_i and ψ'_i . Since Z is freely generated every formula in the left hand side of equation (12) also appears in its right hand side, and vice versa. Hence both formulae $\phi \oplus \sigma$ and $\psi \oplus \sigma$ yield the same class in $Z \cong \mathbb{Z}[\mathcal{F}]$, whence must be \mathcal{I} -isomorphic. \square

4. AFFINE SCHEMES OF FINITE TYPE

All schemes are assumed to be Noetherian, even if we do not always mention this. Let $X = \text{Spec } A$ be an affine Noetherian scheme. By an X -scheme, we mean a separated scheme Y together with a morphism of finite type $Y \rightarrow X$. Hence, affine X -schemes are in one-one correspondence with finitely generated A -algebras. We call a scheme Y a *variety* if it is reduced (but not necessarily irreducible).

Fix an affine Noetherian scheme $X = \text{Spec } A$. Let \mathcal{L}_A be the language of A -algebras in the signature consisting of two binary operations $+$ and \cdot , plus constant symbols for each element in A . A formula in this language will simply be called an *A -algebra formula*. By a *schemic* formula in \mathcal{L}_A , we mean a finite conjunction of formulae $f(x) = 0$, where $f \in A[x]$ and x is a finite tuple of indeterminates. We denote the collection of all schemic formulae by Sch . Let \mathbf{T}_A be the theory of A -algebras, that is to say, the theory whose models are the \mathcal{L}_A -structures that carry the structure of an A -algebra. We also consider some extensions of this theory. As we shall see, and as would not be surprising to an algebraic geometer, it suffices to work with local rings. Being local is a first-order property as it is equivalence with the statement that the sum of any two non-units is again a non-unit, and hence \mathbf{T}_A^{loc} is the theory \mathbf{T}_A to which we adjoin the first-order sentence

$$(\forall x, y) \left((\forall a)(ax \neq 1) \wedge (\forall b)(by \neq 1) \Rightarrow (\forall c)((x + y)c \neq 1) \right).$$

But not only can we restrict to local rings, we may restrict our theory to zero-dimensional algebras. More precisely, let \mathbf{Art}_A be the class of all local A -algebras of finite length as an A -module, for short, the *local A -Artinian algebras*. Unfortunately, \mathbf{Art}_A is not elementary, and hence its theory will have models that are not Artinian local rings. Nonetheless, as observed earlier, whenever we have to verify an equivalence or an isomorphism modulo this theory, it suffices to check this on the rings in \mathbf{Art}_A . Finally, the “classical” theory is recovered from looking at the theory \mathbf{ACF}_A , consisting of all algebraically closed fields that are A -algebras. Instead of writing \mathcal{L}_A and \mathbf{Art}_A , we also may write \mathcal{L}_X or \mathbf{Art}_X , when we take a more geometrical point of view. Similarly, given an A -algebra B and a schemic formula ϕ , we call $\phi(B)$ sometimes the definable subset of $Y := \text{Spec } B$ given by ϕ , and denote it $\phi(Y)$.

4.1. Theorem. *Let A be a Noetherian ring, x an n -tuple of indeterminates, and \mathbb{A}_A^n the affine scheme $\text{Spec}(A[x])$. There is a one-one correspondence between the following three sets:*

- (1) *the set of \mathbf{T}_A -equivalence classes of schemic formulae of arity n ;*
- (2) *the set of \mathbf{Art}_A -equivalence classes of schemic formulae of arity n ;*
- (3) *the set of all ideals in $A[x]$;*
- (4) *the set of all closed subschemes of \mathbb{A}_A^n .*

Proof. The one-one correspondence between (3) and (4) is of course well-known: to an ideal $I \subseteq A[x]$ one associates the closed subscheme $\text{Spec}(A[x]/I)$. Let ϕ be the schemic formula $f_1(x) = \dots = f_s(x) = 0$, with $f_i \in A[x]$, and let $I(\phi) := (f_1, \dots, f_s)A[x]$. Suppose ψ is another schemic formula in the free variables x which is \mathbf{T}_A -equivalent to ϕ . We need to show that $I(\phi) = I(\psi)$. Let $C_\phi := A[x]/I(\phi)$. Since x satisfies ψ in the A -algebra C_ψ , it must also satisfy ϕ in C_ψ , by definition of equivalence. This means that all $f_i(x)$ are zero in C_ψ , that is to say, $f_i \in I(\psi)$. Hence $I(\phi) \subseteq I(\psi)$. Reversing the argument then shows that both ideals are equal. Conversely, if both ideals are the same, then writing the f_i in terms of the generators of $I(\psi)$ shows that any solution to ψ , in any A -algebra, is also a solution to ϕ , and vice versa. Hence the two formulas are equivalent.

So remains to show the equivalence of (2) with the remaining conditions. One direction is trivial, so assume ϕ and ψ are \mathbf{Art}_A -equivalent, that is to say, $\phi(R) = \psi(R)$ for all local A -Artinian algebras R . Towards a contradiction, assume $I := I(\phi)$ and $J := I(\psi)$ are different ideals in $B := A[x]$. Hence, there exists a maximal ideal $\mathfrak{m} \subseteq B$ such that $IB_{\mathfrak{m}} \neq JB_{\mathfrak{m}}$. Moreover, by Krull's Intersection Theorem, there exists an n such that $IR \neq JR$, where $R := B_{\mathfrak{m}}/\mathfrak{m}^n B_{\mathfrak{m}} = B/\mathfrak{m}^n$. Let a be the image of the n -tuple x in R . We have $a \in \phi(R/IR)$, since each $f_i(a) = 0$ in R/IR . Since R/IR has finite length as an A -module, $\phi(R/IR) = \psi(R/IR)$. Hence, $g(a) = 0$ in R/IR , for any $g \in J$, showing that $g \in IR$ whence $JR \subseteq IR$. Switching the role of I and J , the latter inclusion is in fact an equality, contradicting the choice of R . \square

From the proof of Theorem 4.1, we see that the ideal $I(\phi)$ associated to a schemic formula ϕ only depends on the \mathbf{Art}_A -equivalence class of ϕ . We denote the affine scheme corresponding to ϕ by Y_ϕ , that is to say, $Y_\phi := \text{Spec}(A[x]/I(\phi))$.

4.1.1. Base change. Let A' be an A -algebra, that is to say, a morphism $X' := \text{Spec}(A') \rightarrow X := \text{Spec}(A)$. We may assign to each A -algebra B its scalar extension $A' \otimes_A B$, or in terms of affine schemes, $Y = \text{Spec}(B)$ yields by base change the affine X' -scheme $X' \times_X Y$. In terms of formulae, if ϕ is the schemic \mathcal{L}_A -formula defining Y , then we may view ϕ also as an $\mathcal{L}_{A'}$ -formula. As such, it defines the base change $X' \times_X Y$.

Under the one-one correspondence of Theorem 4.1, schemic sentences correspond to ideals of A . More precisely, if σ is the schemic sentence $a_1 = \dots = a_s = 0$ with $a_i \in A$, and $\mathfrak{a} := (a_1, \dots, a_s)A$ the corresponding ideal, then in a model C of \mathbf{T}_A , the interpretation of σ is either the empty set, in case $\mathfrak{a}C \neq 0$, or the singleton $\{\emptyset\}$, in case $\mathfrak{a}C = 0$.

Note that the previous result is false for non-schemic formulae. For instance, if $\phi(x) := (x^2 = x)$ and $\psi(x) := (x = 0) \vee (x = 1)$, then $\phi \Rightarrow \psi$ in the theory \mathbf{Art}_A (since local rings only have trivial idempotents), but not in \mathbf{T}_A (take, for instance, as model $C := A[x]/(x^2 - x)A[x]$). In fact, the schemic formula $y = x$ defines a morphism $\phi \rightarrow \psi$ in \mathbf{Art}_A , but not in \mathbf{T}_A (take again C as the model; see also Example 4.3 below).

Unless stated explicitly otherwise, we will from now on assume that the underlying theory is \mathbf{Art}_A .

Disjoint sums and unions. Although we have the general construction of a disjoint sum \oplus of two schemic formula, the result is no longer a schemic formula. To this end, we define the *disjoint union* of two schemic formulae ϕ and ψ as follows. Let n be the maximum of the arities of ϕ and ψ , and put $z := v_{n+1}$. Let $I(\phi)$ and $I(\psi)$ be the respective ideals of ϕ and ψ in $\mathbb{Z}[x]$, with $x = (x_1, \dots, x_n)$, and put

$$\mathfrak{a} := ((1 - z)I(\phi), zI(\psi), z(z - 1))\mathbb{Z}[x, z].$$

Then the *disjoint union* of ϕ and ψ is the schemic formula $\phi \sqcup \psi$ given by the ideal \mathfrak{a} , that is to say, the conjunction of all equations $(z - 1)f = 0$ and $zg = 0$, with $f \in I(\phi)$ and $g \in I(\psi)$, together with $z(z - 1) = 0$.

4.2. Lemma. *Given schemic formulae ϕ and ψ , their disjoint union $\phi \sqcup \psi$ is \mathbf{Art}_A -equivalent to their disjoint sum $\phi \oplus \psi$.*

Proof. It suffices to show that both formulae define the same subset in any local A -Artinian algebra R . Let ϕ' and ψ' be the formulae $\phi(x) \wedge (z = 0)$ and $\psi(x) \wedge (z = 1)$ respectively, where $(x, z) = (v_1, \dots, v_{n+1})$ as above. In particular, $\phi \oplus \psi = \phi' \vee \psi'$. Suppose $(a, c) \in (\phi \sqcup \psi)(R)$. Since R is local, its only idempotents are 0 and 1. Hence, since $c^2 = c$, we may assume, upon reversing the role of ϕ and ψ if necessary, that $c = 0$. In particular, $(1 - c)f(a) = f(a) = 0$ in R , for all $f \in I(\phi)$, showing that $a \in \phi(R)$, whence $(a, c) \in (\phi' \vee \psi')(R)$. Conversely, suppose $(\phi' \vee \psi')(a, c)$ holds in R , so that one of the disjuncts is true in R , say, $\phi'(a, c)$. In particular, $0 = c = c(c - 1)$ and $f(a) = 0$ for all $f \in I(\phi)$, showing that $(a, c) \in (\phi \sqcup \psi)(R)$. \square

4.3. Example. All we needed from the models of \mathbf{Art}_A was that they were local. However, the result is false in \mathbf{T}_A : for instance, let λ be the schemic (Lefschetz) formula corresponding to the zero ideal in $A[x]$, with x a single variable. Then $\lambda \oplus \lambda$ is the formula $(z = 0) \vee (z = 1)$, whereas $\lambda \sqcup \lambda$ is the formula $z^2 - z = 0$ (with the usual primary form assumption that $x = v_1$ and $z = v_2$). However, in the A -algebra $C := A[t]/(t^2 - t)A[t]$ these formulae define different subsets: $(\lambda \oplus \lambda)(C)$ is the subset $C \times \{0, 1\}$, whereas $(\lambda \sqcup \lambda)(C)$ contains $(0, t)$.

Immediately from Lemma 4.2, we get:

4.4. Lemma. *If ϕ and ψ are schemic formulae with corresponding affine schemes Y_ϕ and Y_ψ , then $\phi \sqcup \psi$ corresponds to the disjoint union $Y_\phi \sqcup Y_\psi$.* \square

4.5. Lemma. *For $X := \text{Spec } A$ an affine Noetherian scheme, ϕ a schemic formula defining an affine X -scheme $Y := Y_\phi$, and B an A -algebra, the definable subset $\phi(B)$ is in one-one correspondence with $\text{Mor}_X(\text{Spec } B, Y)$, the set of B -rational points on Y over X .*

Proof. Put $Z := \text{Spec } B$. Recall that $\text{Mor}_X(Z, Y)$ consists of all morphisms of schemes $Z \rightarrow Y$ over X . Such a map is uniquely determined by an A -algebra homomorphism $A[x]/I(\phi) \rightarrow B$, and this in turn, is uniquely determined by the image b of x in B . Since b is therefore a solution of all $f \in I(\phi)$, it satisfies ϕ . Conversely, any tuple $b \in \phi(B)$ induces an A -algebra homomorphism $A[x]/I(\phi) \rightarrow B$. \square

In fact, we may view ϕ as a functor on the category of A -algebras and as such it agrees with the functor represented by the scheme Y_ϕ over X . More precisely, if $B \rightarrow C$ is an A -algebra homomorphism then the induced map $B^n \rightarrow C^n$ on the Cartesian products maps $\phi(B)$ into $\phi(C)$. On the other hand, the associated map of schemes $\text{Spec } C \rightarrow \text{Spec } B$ induces, by composition, a map $\text{Mor}_X(\text{Spec } B, Y_\phi) \rightarrow \text{Mor}_X(\text{Spec } C, Y_\phi)$. Under the

one-one correspondence in Lemma 4.5, we get a commutative diagram

$$(13) \quad \begin{array}{ccc} \phi(B) & \xrightarrow{\quad} & \phi(C) \\ \downarrow & & \downarrow \\ \text{Mor}_X(\text{Spec } B, Y_\phi) & \xrightarrow{\quad} & \text{Mor}_X(\text{Spec } C, Y_\phi). \end{array}$$

Schemic morphisms. By definition, a *Sch*-morphism $f: \phi \rightarrow \psi$ between \mathcal{L}_A -formulae $\phi(x)$ and $\psi(y)$ is given by a schemic formula $\theta(x, y)$, such that $\theta(B)$ is the graph of a function $f_B: \phi(B) \rightarrow \psi(B)$, for every A -algebra B . In particular, θ is explicit, that is to say, belongs to \mathcal{Expl} , if it is of the form $\bigwedge_{i=1}^m y_i = p_i(x)$, with $p_i \in A[x]$. For each A -algebra B , we denote the global map defined by this explicit formula by $p_B: \mathbb{A}_B^n \rightarrow \mathbb{A}_B^m$, where $p = (p_1, \dots, p_m)$. In particular, $f_B: \phi(B) \rightarrow \psi(B)$ is then the restriction of p_B .

4.6. Proposition. *Every schemic morphism between schemic formulae is explicit.*

Proof. Let us first prove this modulo \mathbf{T}_A . Let $f_\theta: \phi \rightarrow \psi$ be a schemic morphism between the schemic formulae $\phi(x)$ and $\psi(y)$, with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$. Let $C := A[x]/I(\phi)$, where $I(\phi)$ is the ideal associated to ϕ under the equivalence given in Theorem 4.1, that is to say, the ideal generated by the equations that make out the schemic formula ϕ . Let a denote the n -tuple in C given by the image of the variables x in C . In particular, $a \in \phi(C)$, and hence $f_C(a) \in \psi(C)$. Let p be a tuple of polynomials in $A[x]$ whose image is the tuple $f_C(a)$ in C . Suppose θ is the conjunction of formulae $h_i(x, y) = 0$, with $h_i \in A[x, y]$, for $i = 1, \dots, s$, and ψ is the conjunction of formulae $g_j(y) = 0$ with $g_j \in A[y]$, for $j = 1, \dots, t$. Put $\tilde{h}_i(x) := h_i(x, p(x))$ and $\tilde{g}_j(x) := g_j(p(x))$. Since both $\theta(a, p(a))$ and $\psi(p(a))$ hold in C , all \tilde{h}_i and \tilde{g}_j belong to $I(\phi)$.

Therefore, if B is an arbitrary A -algebra, and $b \in \phi(B)$, then all $\tilde{h}_i(b)$ and $\tilde{g}_j(b)$ are zero. In particular, $p(b)$ lies in $\psi(B)$ and $\theta(b, p(b))$ holds in B . By condition (2) in the definition of a morphism, $f_B(b)$ must be equal to $p(b)$. Hence we showed that $\theta(x, y)$ is isomorphic to the explicit formula $\gamma(x, y)$ given as the conjunction of all $y_i = p_i(x)$. By Theorem 4.1, the two schemic formulae θ and γ being \mathbf{T}_A -equivalent, are then also \mathbf{Art}_A -equivalent, showing that f is explicit modulo the latter theory. \square

4.7. Example. Not every schemic morphism is explicit. For instance, let $\phi(x_1, x_2)$ be the formula $x_1 x_2 = 1$ and let $\psi(x_1) := (\exists x_2) \phi(x_1, x_2)$. Then the formula $(x_1 = y_1) \wedge (x_1 y_2 = 1)$ is morphic and yields a schemic morphism $f: \psi(x_1) \rightarrow \phi(y_1, y_2)$, but this is not explicit, since $1/x_1$ is not a term (polynomial). Put differently, for any A -algebra B , the map $f_B: \psi(B) \rightarrow \phi(B): b \mapsto (b, 1/b)$ is not induced by any total map $\mathbb{A}_B^1 \rightarrow \mathbb{A}_B^2$. Note that f is even a bijection, and its inverse $\phi(x_1, x_2) \rightarrow \psi(y_1)$ is given by the explicit formula $\theta(x_1, x_2, y_1) := \langle y_1 = x_1 \rangle$. In particular, the latter explicit bijection is not an \mathcal{Expl} -isomorphism, since $\text{inv}(\theta)(x_1, y_1, y_2) = \langle y_1 = x_1 \rangle$ is not explicit, as it does not contain a conjunct of the form $y_2 = g(x_1)$ with g a polynomial (as pointed out, $1/x_1$ is not a term). By Lemma 3.2, however, f is a *Sch*-isomorphism.

4.8. Corollary. *There is a one-one correspondence between schemic morphisms $\phi \rightarrow \psi$ modulo \mathbf{Art}_A and morphisms $Y_\phi \rightarrow Y_\psi$ of schemes over X .*

Proof. If $\phi \rightarrow \psi$ is a schemic morphism, then by Proposition 4.6, there exists a tuple of polynomials p such that $\phi(B) \rightarrow \psi(B)$ is given by the base change of the polynomial map $p: \mathbb{A}_A^n \rightarrow \mathbb{A}_A^m$. By construction, the restriction of p to the closed subscheme Y_ϕ of \mathbb{A}_A^n has image inside the closed subscheme $Y_\psi \subseteq \mathbb{A}_A^m$, and hence induces a morphism $Y_\phi \rightarrow Y_\psi$ over X . Conversely, any morphism $Y_\phi \rightarrow Y_\psi$ is easily seen to induce a morphism $\phi \rightarrow \psi$ under the one-one correspondence given by Lemma 4.5. \square

Proposition 4.6 shows that Sch is compositionally closed relative to Sch .

4.9. Corollary. *For each Noetherian ring A , there is a one-one correspondence between Sch -isomorphism classes of schemic formulae modulo \mathbf{Art}_A , and isomorphism classes of affine schemes of finite type over A .*

Proof. By Theorem 4.1, a schemic formula ϕ corresponds to a scheme Y_ϕ of finite type over $X := \text{Spec } A$, and by Corollary 4.8, a schemic definable map $\phi \rightarrow \psi$ induces a morphism $Y_\phi \rightarrow Y_\psi$ of schemes over X . Using this correspondence, one checks that an isomorphism of formulae corresponds to an isomorphism of schemes over X . \square

Recall that a *pp-formula* (positive primitive formula) is the projection of a schemic formula, that is to say, a formula of the form $(\exists y)\psi(x, y)$, where $\psi(x, y)$ is a schemic formula. To emphasize that pp-formulae represent projections, we will also denote a pp-formula as

$$\text{Im}(\psi)(x) := (\exists y)\psi(x, y).$$

It follows from Lemma 3.1 that the collection of all pp-formulae is compositionally closed.

4.10. Corollary. *If ϕ is a schemic formula and θ a pp-morphism $f_\theta: \phi \rightarrow \psi$, then θ is explicit.*

Proof. By the same argument as in the proof of Proposition 4.6, it suffices to show this modulo \mathbf{T}_A . Let us first show this result for ψ a pp-formula. We can repeat the proof of Proposition 4.6, up to the point that we introduced the polynomials \tilde{h}_i and \tilde{g}_j . Instead, $\theta(x, y)$ is of the form $(\exists z)\zeta(x, y, z)$, with ζ a schemic formula in the variables x , y , and z , and likewise, ψ is of the form $\text{Im}(\gamma) = (\exists z)\gamma(x, z)$, with γ a schemic formula in the variables x and z . Let $I(\zeta)$ and $I(\gamma)$ be generated respectively by polynomials $h_i(x, y, z) \in A[x, y, z]$ and $g_j(x, z) \in A[x, z]$, for $i = 1, \dots, s$ and $j = 1, \dots, t$. Since $\theta(a, p(a))$ and $\psi(p(a))$ hold in C , we can find tuples of polynomials $g, q \in A[x]$ so that $\zeta(a, p(a), g(a))$ and $\gamma(p(a), q(a))$ hold in C , implying that all $\tilde{h}_i(x) := h_i(x, p(x), g(x))$ and $\tilde{g}_j(x) := g_j(p(x), q(x))$ lie in $I(\phi)$. Hence in an arbitrary A -algebra B , we have for every $b \in \phi(B)$, that $\tilde{h}_i(b)$ and $\tilde{g}_j(b)$ are all zero. This proves $p(b) \in \psi(B)$ and $(b, p(b)) \in \theta(B)$, and we can now finish the proof as above for ψ a pp-formula.

For the general case, let ψ be arbitrary, and define the pp-formula

$$\text{Im}(\phi \wedge \theta)(y) := (\exists x)\phi(x) \wedge \theta(x, y).$$

In order to show that θ defines a morphism $\phi \rightarrow \text{Im}(\phi \wedge \theta)$, we may verify this in an arbitrary A -algebra B . Let $c \in \phi(B)$ and put $b := f_B(c)$. Hence $\theta(c, b)$, whence $\text{Im}(\phi \wedge \theta)(b)$, holds, showing that f_B maps $\phi(B)$ inside $\text{Im}(\phi \wedge \theta)(B)$. On the other hand, if $b \in \text{Im}(\phi \wedge \theta)(B)$, then there exists $c \in \phi(B)$ such that $\theta(c, b)$ holds, and hence $b = f_B(c) \in \psi(B)$. Hence, the implication $\text{Im}(\phi \wedge \theta) \Rightarrow \psi$ is explicit. Moreover, by our previous argument, the morphism $\phi \rightarrow \text{Im}(\phi \wedge \theta)$ is explicit, and therefore, so is the composition $\phi \rightarrow \text{Im}(\phi \wedge \theta) \rightarrow \psi$, as we wanted to show. \square

4.11. Corollary. *If ϕ is a pp-formula and $f: \phi \rightarrow \psi$ a pp-morphism, then we can lift f to an explicit morphism. More precisely, if $\phi = \text{Im}(\gamma)$ with γ a schemic formula, then there exist explicit morphisms $p: \gamma \rightarrow \phi$ and $\tilde{f}: \gamma \rightarrow \psi$ with p surjective, yielding a commutative diagram*

$$(14) \quad \begin{array}{ccc} & \gamma & \\ p \swarrow & & \searrow \tilde{f} \\ \phi & \xrightarrow{f} & \psi. \end{array}$$

Proof. Let $p: \gamma \rightarrow \phi$ be the projection map, that is to say, the explicit morphism given by $y = x$. Hence the composition $\tilde{f} := f \circ p$ is a pp-morphism. By Corollary 4.10, this is an explicit morphism, as we wanted to show. \square

Zariski closure of a formula. Given a formula ψ in \mathcal{L}_A , we define its *Zariski closure* $\bar{\psi}$ as follows. Suppose ψ has arity n . Let I be the sum of all $I(\phi)$, with $\phi \in \text{Sch}_n$ such that $\psi \Rightarrow \phi$ holds in \mathbf{T}_A , and let $\bar{\psi}$ be a schemic formula corresponding to the ideal I . Hence, by Theorem 4.1, the Zariski closure is defined up to \mathbf{Art}_A -equivalence. Moreover, it satisfies the following universal property: $\psi \Rightarrow \bar{\psi}$, and if ϕ is a schemic formula such that $\psi \Rightarrow \phi$, then $\bar{\psi} \Rightarrow \phi$. From the definition, it follows that $\bar{\psi}$ is equivalent to the infinite conjunction of all $\phi \in \text{Sch}_n$ such that $\psi \Rightarrow \phi$.

4.12. Lemma. *Every explicit morphism $f: \psi \rightarrow \psi'$ of \mathcal{L}_A -formulae extends to an explicit morphism $\bar{f}: \bar{\psi} \rightarrow \bar{\psi}'$.*

Proof. Let θ be an explicit formula defining the morphism $f_\theta: \psi \rightarrow \psi'$, and let $p: \mathbb{A}_A^n \rightarrow \mathbb{A}_A^m$ be the total map defined by θ . Again, by Theorem 4.1, we may work modulo \mathbf{T}_A . Hence for any A -algebra B , the induced map f_B is simply the restriction of the base change p_B of p . So remains to show that p_B maps $\bar{\psi}(B)$ inside $\bar{\psi}'(B)$. Let $\phi(x)$ be the schemic formula $\bar{\psi}'(p(x))$ obtained by substituting p for y . It follows that $\psi(B) \subseteq \phi(B)$. Since this holds for all B , we get $\psi \Rightarrow \phi$ and hence by the universal property of Zariski closure, $\bar{\psi} \Rightarrow \phi$. It is now easy to see that this means that p_B maps $\bar{\psi}(B)$ inside $\bar{\psi}'(B)$. \square

The non-explicit, schemic map from Example 4.7 does not extend to the Zariski closure of ψ , as $\bar{\psi}(x_1) = (x_1 = x_1)$.

4.13. Corollary. *If two \mathcal{L}_A -formulae ψ and ψ' are $\mathcal{E}xpl$ -isomorphic, then so are their Zariski closures $\bar{\psi}$ and $\bar{\psi}'$.*

Proof. Let θ and $\text{inv}(\theta)$ be explicit formulae defining respectively $f: \psi \rightarrow \psi'$ and its inverse $g: \psi' \rightarrow \psi$. Applying Lemma 4.12 to both morphisms yields explicit morphisms $\bar{f}: \bar{\psi} \rightarrow \bar{\psi}'$ and $\bar{g}: \bar{\psi}' \rightarrow \bar{\psi}$. Moreover, since the compositions $g \circ f: \psi \rightarrow \psi$ and $f \circ g: \psi' \rightarrow \psi'$ are both identity morphisms, so must their Zariski closures be, and it is not hard to see that these are $\bar{g} \circ \bar{f}$ and $\bar{f} \circ \bar{g}$ respectively. Hence, we showed that \bar{f} and \bar{g} are each others inverse. \square

The Zariski closure of a formula can in general be hard to calculate. Here is a simple example: if F is an algebraically closed field and $f, g \in F[x]$ are relatively prime polynomials in a single variable, then the Zariski closure of $\psi := (f = 0) \vee (g = 0)$ is the formula $f g = 0$. It is clear that $\psi \Rightarrow (f g = 0)$. To show that it satisfies the universal

property for Zariski closures, let ϕ be any spheric formula implied by ψ . We need to show that $I(\phi) \subseteq fgF[x]$. We may reduce therefore to the case that ϕ is the formula $h = 0$, and hence we have to show that any root of f or g in F is also a root of h of at least the same multiplicity. By the Nullstellensatz, and after a translation, it suffices to prove that if 0 is a root of f of multiplicity e , then it is also a root of h of multiplicity e . Write $f(x) = x^e \tilde{f}(x)$, for some $\tilde{f} \in F[x]$. Let $B := F[x]/fgF[x]$, and put $b := x\tilde{f}g \in F[x]$. Since $f(b) = x^e \tilde{f}^e g^e \tilde{f}(b)$, it is zero in B , that is to say, $b \in \psi(B)$. Hence, by assumption, $b \in \phi(B)$, that is to say, $h(b) = 0$ in B . In particular, $h(b)$ is divisible by x^e in $F[x]$, and writing out h as polynomial in x then easily implies that h itself must be divisible by x^e . Hence $x = 0$ is a root of h of multiplicity e , as we wanted to show. We expect that this result holds true in far greater generality: is it the case that the Zariski closure of a disjunction $\phi_1 \vee \phi_2 \vee \dots \vee \phi_s$ of spheric formulae ϕ_i is the spheric formula corresponding to the ideal $I(\phi_1) \cap I(\phi_2) \cap \dots \cap I(\phi_s)$?

5. THE SCHEMIC GROTHENDIECK RING

Let $X := \text{Spec } A$ be an affine, Noetherian scheme.

The classical Grothendieck ring. Before we discuss our generalization to schemes, let us first study the classical case. To this end, we must work in the theory \mathbf{ACF}_X , the theory of algebraically closed fields having the structure of an A -algebra. We have the following analogue of Theorem 4.1.

5.1. Theorem. *Let A be a Noetherian ring, x an n -tuple of indeterminates, and \mathbb{A}_A^n the affine scheme $\text{Spec}(A[x])$. There is a one-one correspondence between the following three sets:*

- (1) *the set of \mathbf{ACF}_A -equivalence classes of spheric formulae of arity n ;*
- (2) *the set of radical ideals in $A[x]$;*
- (3) *the set of reduced subschemes of \mathbb{A}_A^n .*

Proof. The one-one correspondence between the last two sets is again classical. Let ϕ and ψ be spheric formulae in the free variables x . Assume first that ϕ and ψ are \mathbf{ACF}_A -equivalent. We need to show that $I(\phi)$ and $I(\psi)$ have the same radical. Suppose not, so that there exists a prime ideal $\mathfrak{p} \subseteq A[x]$ containing exactly one of these ideals, say, $I(\phi)$, but not the other. Let K be the algebraic closure of $A[x]/\mathfrak{p}$, so that K is a model of \mathbf{ACF}_A , and let a denote the image of x in K . By assumption, we can find $f \in I(\psi)$ such that $f \notin \mathfrak{p}$. In particular, since $f(a) \neq 0$ in K , the tuple a does not belong to $\psi(K) = \phi(K)$. However, for any $g \in I(\phi)$, we have $g \in \mathfrak{p}$ whence $g(a) = 0$ in K , contradiction.

Conversely, if both ideals have the same radical, then each $f \in I(\phi)$ has some power f^N belonging to $I(\psi)$. In particular, if K is a model of \mathbf{ACF}_A and $c \in \psi(K)$, then $f^N(c)$ whence also $f(c)$ vanishes in K , for all $f \in I(\phi)$, showing that $c \in \phi(K)$. This shows that $\psi(K) \subseteq \phi(K)$, and the reverse inclusion follows by the same argument, proving that ϕ and ψ are \mathbf{ACF}_A -equivalent. \square

Let X be a Noetherian affine scheme. We define its *classical Grothendieck ring* to be the Grothendieck ring

$$\mathbf{Gr}(X^{\text{var}}) := \mathbf{Gr}_{\mathcal{E}xpl}^{\mathbf{ACF}_X}(\mathcal{QF}),$$

that is to say, the ring obtained by killing the ideal of all scissor relations and all $\mathcal{E}xpl$ -isomorphism relations in the free Abelian group on all classes of quantifier free formulae modulo \mathbf{ACF}_X . The analogue of Corollary 4.9 holds, showing that the set of $\mathcal{E}xpl$ -isomorphism classes of spheric formulae modulo \mathbf{ACF}_X is in one-one correspondence

with the set of isomorphism classes of reduced affine X -schemes. Moreover, Corollary 3.9 applies with $\mathcal{G} = \text{Sch}$, so that the classes of schemic formulae generate this Grothendieck ring. Hence, we showed the first assertion of:

5.2. Corollary. *If F is an algebraically closed field, then $\mathbf{Gr}(F^{\text{var}})$ is the Grothendieck ring $K_0(F)$ obtained by taking the free Abelian group on isomorphism classes of varieties and killing all scissor relations. Moreover, if F has characteristic zero, then it is also equal to the full Grothendieck ring $\mathbf{Gr}(\mathbf{ACF}_F)$ of the theory \mathbf{ACF}_F .*

Proof. To prove the last assertion, observe that \mathbf{ACF}_F has quantifier elimination, and therefore the full Grothendieck ring $\mathbf{Gr}(\mathbf{ACF}_F)$ is generated by the classes of quantifier free formulae. The only issue is the nature of isomorphism. In view of Corollary 3.9, it suffices to show that if $f: Y \rightarrow X$ is an \mathbf{ACF}_F -isomorphism of affine F -schemes, then $[X] = [Y]$ in $\mathbf{Gr}(F^{\text{var}})$. By [13, ?], we can find a constructible partition $Y = Y_1 \cup \dots \cup Y_s$ of Y , such that each restriction $f|_{Y_i}$ is an explicit isomorphism (note that we need characteristic zero to avoid having to take p -th roots). Hence $[Y_i] = [f(Y_i)]$ in $\mathbf{Gr}(F^{\text{var}})$, and the result now follows since $[Y]$ and $[X]$ are the respective sums of all $[Y_i]$ and all $[f(Y_i)]$. \square

The schemic Grothendieck ring. Let \mathcal{QF} be the Boolean closure of Sch , that is to say, the lattice of *quantifier free formulae*. We define the *schemic Grothendieck ring* of X as

$$\mathbf{Gr}(X^{\text{sch}}) := \mathbf{Gr}_{\text{Sch}}^{\mathbf{Art}_X}(\mathcal{QF}).$$

As before, we denote the class of a formula by $[\phi]$, and in case Y is an affine X -scheme, we also write $[Y]$ for the class of its defining schemic formula given by Corollary 4.9, and henceforth, identify both. In particular, the base scheme X corresponds to the class of the sentence \top , which we will denote simply by 1.

5.3. Lemma. *Let X be an affine, Noetherian scheme. Any element in $\mathbf{Gr}(X^{\text{sch}})$ is of the form $\sum_{i=1}^s n_i [Y_i]$, for some integers n_i , and some affine X -schemes Y_i . Alternatively, we may write any element as $[Z] - [Z']$, for Z and Z' affine X -schemes.*

Proof. Both statements follow immediately from Theorem 4.1 and Corollary 3.9, since Sch is closed, modulo \mathbf{Art}_X , under conjunctions, and, by Lemma 4.2, disjoint sums. \square

In particular, the natural morphism $\mathbf{Gr}_{\text{Sch}}^{\mathbf{Art}_X}(\text{Sch}) \rightarrow \mathbf{Gr}(X^{\text{sch}})$ is surjective. Let us call two X -schemes Y and Y' *stably isomorphic*, if there exists an affine X -scheme Z such that $Y \sqcup Z$ and $Y' \sqcup Z$ are isomorphic over X . A priori this is a weaker equivalence relation than the isomorphism relation, but for affine Noetherian schemes, it is the same, as we will discuss in Appendix 11.

5.4. Theorem. *Two affine X -schemes Y and Y' are isomorphic if and only if their classes $[Y]$ and $[Y']$ in $\mathbf{Gr}(X^{\text{sch}})$ are the same.*

Proof. By Theorem 3.11, the defining schemic formulae ϕ and ϕ' of respectively Y and Y' are stably Sch -isomorphic, that is to say,

$$\phi \oplus \psi \cong_{\text{Sch}} \phi' \oplus \psi,$$

for some quantifier free formula ψ . By Lemma 3.10, we may replace ψ by any formula implied by it, whence, in particular, by its Zariski closure. In conclusion, we may assume ψ is schemic. By Lemma 4.2, disjoint sum and union are \mathbf{Art}_X -equivalent. Hence, if Z denotes the affine scheme defined by ψ , then $Y \sqcup Z \cong Y' \sqcup Z$ by Corollary 4.8, that is to say, Y and Y' are stably isomorphic, whence isomorphic, by Theorem 11.4. \square

By definition of the Lefschetz class, the class of $\mathbb{A}_X^1 := \text{Spec}(A[x])$, with x a single indeterminate, is \mathbb{L} . In particular, we get the following generalization of Lemma 3.6:

5.5. Lemma. *If Y is an affine X -scheme, then $[\mathbb{A}_Y^n] = \mathbb{L}^n \cdot [Y]$ in $\mathbf{Gr}(X^{\text{sch}})$.* \square

6. THE PP-GROTHENDIECK RING OF X

Theorem 5.4 essentially says that $\mathbf{Gr}(X^{\text{sch}})$ creates no further relations among X -schemes. To obtain non-trivial relations among classes of X -schemes, we will now work in a larger class of formulae. It turns out that pp-formulae are sufficiently general to accomplish this. To obtain the greatest amount of versatility, rather than working within the semi-lattice of pp-formulae, we will work in the Boolean lattice \mathcal{PP} , given as the Boolean closure of all pp-formulae, that is to say, all finite disjunctions of pp-formulae and their negations. Recall that we write $\text{Im}(\phi)(x)$ to denote the pp-formula $(\exists y)\phi(x, y)$, for ϕ a schemic formula. We define the *pp-Grothendieck ring* of $X = \text{Spec } A$ to be the Grothendieck ring

$$\mathbf{Gr}(X^{\text{pp}}) := \mathbf{Gr}_{\text{Sch}}^{\mathbf{Art}_X}(\mathcal{PP}).$$

By Corollary 3.5, we have a natural ring homomorphism $\mathbf{Gr}(X^{\text{sch}}) \rightarrow \mathbf{Gr}(X^{\text{pp}})$. The analogue of Lemma 4.2 holds, and in particular, we can describe the elements of $\mathbf{Gr}(X^{\text{pp}})$ as in Lemma 5.3:

6.1. Proposition. *The disjoint sum of two pp-formulae is \mathbf{Art}_X -equivalent to a pp-formulae. In particular, every element of $\mathbf{Gr}(X^{\text{pp}})$ can be written as a difference $[\psi] - [\psi']$ with ψ and ψ' pp-formulae.*

Proof. Let $\phi(x) := \text{Im}(\phi_0)$ and $\psi(x) := \text{Im}(\psi_0)$ be two pp-formulae, with $\phi_0(x, y)$ and $\psi_0(x, y)$ schemic formulae. As in the proof of Lemma 5.3, one easily shows that the disjoint union $\phi \sqcup \psi$ is \mathbf{Art}_X -equivalent with $(\exists y)(\phi_0 \sqcup \psi_0)$. The second assertion now follows from this and Corollary 3.9. \square

Assume $Y = \text{Spec } B$ is an affine, Noetherian scheme. Let Θ_Y^{aff} be the collection of all affine opens of Y . We view Θ_Y^{aff} as a semi-lattice with \wedge given by intersection. In general, the union of affine opens need not be affine, so that we cannot define \vee on Θ_Y^{aff} , and therefore, it is only a sub-semi-lattice of the lattice Θ_Y of all opens of Y . Recall that for a finite, open affine covering $\mathcal{U} = \{U_1, \dots, U_n\}$ of Y , we have a scissor relation

$$Y = S_n(U_1, \dots, U_n)$$

in $\mathbf{Sciss}(\Theta_Y)$. The map $\Theta_Y^{\text{aff}} \rightarrow \mathbf{Gr}(X^{\text{pp}})$, sending an affine open $U \subseteq Y$ to its class $[U]$ in $\mathbf{Gr}(X^{\text{pp}})$, extends to an additive map $\mathbb{Z}[\Theta_Y^{\text{aff}}] \rightarrow \mathbf{Gr}(X^{\text{pp}})$ (note that this map is not multiplicative since multiplication on the former is different from that on the latter). We will show in Corollary 6.5 below that it in fact induces a ring homomorphism $\mathbf{K}_{\cong_X}(\Theta_Y^{\text{aff}}) \rightarrow \mathbf{Gr}(X^{\text{pp}})$. Among the members of Θ_Y^{aff} are the *basic* open subsets $D(f) = \text{Spec}(B_f)$, with f a non-nilpotent element of B (so that in particular, a basic open subset is never empty). Note that if $D(f)$ is a basic open, and $U \subseteq Y$ and affine open, then $D(f) \cap U$ is the basic open $D(f|_U)$ in U .

6.2. Proposition. *Let X be a Noetherian scheme, and Y an affine X -scheme. For every finite covering $\{D_1, \dots, D_n\}$ of Y by basic open subsets, we have an identity $[Y] = [S_n(D_1, \dots, D_n)]$ in $\mathbf{Gr}(X^{\text{pp}})$.*

Proof. Let $A := \mathcal{O}_X$, and let $\phi(x)$ be a schemic formula defining Y , that is to say, $Y = \text{Spec } B$ with $B := A[x]/I(\phi)$. By definition of basic subset, there exist $f_i \in A[x]$ so that $D_i = D(f_i) = \text{Spec}(B_{f_i})$. The fact that the D_i cover Y is equivalent with $(f_1, \dots, f_n)B$ being the unit ideal. Hence, we can find $u_i \in A[x]$ and $m \in I(\phi)$, such that

$$(15) \quad \sum_{i=1}^n u_i f_i + m = 1.$$

Consider the following formulae in the variables x and z :

$$\begin{aligned} \psi_i(x, z) &:= \phi(x) \wedge (f_i(x)z = 1) \\ \tilde{\psi}_i(x) &:= \text{Im}(\psi_i)(x) = (\exists z)\psi_i(x, z). \end{aligned}$$

The schemic formula $x = y$ defines a schemic isomorphism $\tilde{\psi}_i(x) \rightarrow \psi_i(y, z)$, for each i . Let $\tilde{\psi}$ be the disjunction of all $\tilde{\psi}_i$. By Proposition 2.3, we have a scissor identity $[\tilde{\psi}] = [S_n(\tilde{\psi}_1, \dots, \tilde{\psi}_n)]$ in $\mathbf{Gr}(X^{\text{pp}})$. Since ψ_i and $\tilde{\psi}_i$ are *Sch*-isomorphic, so are any of their conjunctions, and hence $[S_n(\tilde{\psi}_1, \dots, \tilde{\psi}_n)] = [S_n(\psi_1, \dots, \psi_n)]$ in $\mathbf{Gr}(X^{\text{pp}})$. By definition of basic subset, ψ_i is the defining schemic formula of D_i , and hence $[S_n(\psi_1, \dots, \psi_n)]$ is equal to the class of $S_n(D_1, \dots, D_n)$ in $\mathbf{Gr}(X^{\text{pp}})$. So remains to show that $Y = [\phi] = [\tilde{\psi}]$.

To this end, we have to show that $\tilde{\psi}(R) = \phi(R)$ in any Artinian local A -algebra R . The direct inclusion is immediate, so assume $a \in \phi(R)$. From (15) and the fact that $m(a) = 0$, we get

$$\sum_{i=1}^n u_i(a) f_i(a) = 1.$$

Since R is local, one of these terms must be a unit, say, the first one. Therefore, there exists $b \in R$ such that $b f_1(a) = 1$, and hence $a \in \tilde{\psi}_1(R) \subseteq \tilde{\psi}(R)$, as we needed to show. \square

6.3. Remark. As already observed, this shows that we have an additive map $\mathbf{Gr}(\Theta_Y^{\text{aff}}) \rightarrow \mathbf{Gr}(X^{\text{pp}})$. A cautionary note: although one informally states that a basic open subset $D := D(f)$ is given by the equation $f \neq 0$ in $\text{Spec}(B)$ (as it is the complement of the closed subset given by the equation $f = 0$), this is not correct from the point of view of formulae. As we saw in the above proof, D is defined by the schemic formula $\psi(x, z) := \langle f(x)z = 1 \rangle$, or alternatively, by its projection, the pp-formula $\text{Im}(\psi)(x) := \langle (\exists z)f(x)z = 1 \rangle$. That this is different from the formula $\langle f \neq 0 \rangle$ is easily checked on an example: let $f(x) := x \in F[x]$, with x a single variable, and compare both formulae in the Artinian local ring $F[T]/T^2 F[T]$ (we will see below that in the latter model, nonetheless, the basic open $D(f)$ is given by the quantifier free (non-schemic) formula $f^2 \neq 0$).

A second issue requiring some care is the difference between conjunctions and intersections. Let $D' := D(f')$ be another basic open in Y , and let $\psi' := \langle f'z = 1 \rangle$ and $\text{Im}(\psi') := \langle (\exists z)f'z = 1 \rangle$ be its respective schemic and pp defining formula. The intersection $D \cap D'$ is again a basic open subset, whence an affine scheme. However, $D \cap D'$ is not defined by the schemic formula $\psi \wedge \psi'$, but by the schemic formula $\langle f f'z = 1 \rangle$. Nonetheless, $D \cap D'$ is defined by the pp-formula $\text{Im}(\psi) \wedge \text{Im}(\psi')$.

6.4. Theorem. *There exists a well-defined map which assigns to any (isomorphism class of an) X -scheme Y an element $[Y]$ in $\mathbf{Gr}(X^{\text{pp}})$ which agrees on affine schemes with*

the class map. Moreover, if $\{U_1, \dots, U_n\}$ is any open covering of an X -scheme Y , then $[Y] = [S_n(U_1, \dots, U_n)]$ in $\mathbf{Gr}(X^{\text{pp}})$.

Proof. We start with proving that the second assertion holds in case Y is affine. So let $\mathcal{U} := \{U_1, \dots, U_s\}$ be an open affine covering of Y , and we need to show that $[Y] = [S(U_1, \dots, U_n)]$ in $\mathbf{Gr}(X^{\text{pp}})$. We induct on the number e of non-basic opens among the U_i . If $e = 0$, then the result holds by Proposition 6.2. So assume $e > 1$. Let U_n be a non-basic open subset, and let $\{D_1, \dots, D_m\}$ be an open covering of U_n by basic opens. Using Lemma 2.1(3), as in the proof of Proposition 2.3, we have an identity

$$(16) \quad S(U_1, \dots, U_{n-1}, D_1, \dots, D_m, U_n) = S(U_1, \dots, U_{n-1}, D_1, \dots, D_m) - S(U_1 \cap U_n, \dots, U_{n-1} \cap U_n, D_1, \dots, D_m) + U_n$$

Since $\{U_1, \dots, U_{n-1}, D_1, \dots, D_m\}$ and $\{U_1 \cap U_n, \dots, U_{n-1} \cap U_n, D_1, \dots, D_m\}$ are open affine coverings of Y and U_n respectively, both containing less than e non-basic open subsets, our induction hypothesis yields

$$[Y] = [S(U_1, \dots, U_{n-1}, D_1, \dots, D_m)] \quad \text{and} \\ [U_n] = [S(U_1 \cap U_n, \dots, U_{n-1} \cap U_n, D_1, \dots, D_m)],$$

in $\mathbf{Gr}(X^{\text{pp}})$. Taking classes of both sides of (16) together with the latter identities, shows that $[Y] = [S(U_1, \dots, U_n, D_1, \dots, D_m)]$ (note that the order in scissor relations is irrelevant). We will now prove by induction on m , that $S(U_1, \dots, U_n, D_1, \dots, D_m)$ and $S(U_1, \dots, U_n)$ have the same class in $\mathbf{Gr}(X^{\text{pp}})$. By Lemma 2.1(3), we have an identity

$$(17) \quad S(U_1, \dots, U_n, D_1, \dots, D_m) = S(U_1, \dots, U_n, D_1, \dots, D_{m-1}) - S(U_1 \cap D_m, \dots, U_n \cap D_m, D_1 \cap D_m, \dots, D_{m-1} \cap D_m) + D_m$$

in $\mathbb{Z}[\Theta_Y^{\text{aff}}]$. By induction on m , we have

$$[S(U_1, \dots, U_n)] = [S(U_1, \dots, U_n, D_1, \dots, D_{m-1})]$$

in $\mathbf{Gr}(X^{\text{pp}})$. On the other hand, since $\{U_i \cap D_m, D_j \cap D_m\}$, for $i = 1, \dots, n$ and $j = 1, \dots, m-1$, is an open affine covering of D_m containing less than e non-basic open subsets (note that $U_n \cap D_m = D_m$), our first induction hypothesis (on e) yields

$$[D_m] = [S(U_1 \cap D_m, \dots, U_n \cap D_m, D_1 \cap D_m, \dots, D_{m-1} \cap D_m)]$$

in $\mathbf{Gr}(X^{\text{pp}})$. Hence, taking classes of both sides of (17) together with the previous two identities, yields the desired conclusion

$$[Y] = [S(U_1, \dots, U_n, D_1, \dots, D_m)] = [S(U_1, \dots, U_n)].$$

We now prove the first assertion for Y an arbitrary X -scheme, that is to say, a (not necessarily affine) separated scheme Y of finite type over X . Let $\mathcal{U} := \{U_1, \dots, U_s\}$ be an open affine covering of Y , and define

$$(18) \quad [Y]_{\mathcal{U}} := [S(U_1, \dots, U_n)].$$

Since Y is separated, each intersection $U := U_{i_1} \cap \dots \cap U_{i_k}$ is again affine, so that $[S(U_1, \dots, U_n)]$ is indeed an element of $\mathbf{Gr}(X^{\text{pp}})$. We want to show that $[Y]_{\mathcal{U}}$, as an element of $\mathbf{Gr}(X^{\text{pp}})$, does not depend on the open affine covering \mathcal{U} . To this end, let \mathcal{V} be a second open affine covering of Y , and we seek to show that $[Y]_{\mathcal{U}} = [Y]_{\mathcal{V}}$ in $\mathbf{Gr}(X^{\text{pp}})$. Replacing \mathcal{V} by $\mathcal{U} \cup \mathcal{V}$ if necessary, we may assume that $\mathcal{U} \subseteq \mathcal{V}$. By induction on the

number of members of \mathcal{V} , we may then reduce to the case that $\mathcal{V} = \mathcal{U} \cup \{V\}$, for some affine open $V \subseteq Y$. By Lemma 2.1(3), we have, in $\mathbb{Z}[\Theta_Y^{\text{aff}}]$, an identity

$$(19) \quad S(U_1, \dots, U_n, V) = S(U_1, \dots, U_n) - S(U_1 \cap V, \dots, U_n \cap V) + V.$$

Since $\{U_1 \cap V, \dots, U_n \cap V\}$ is an open affine covering of the affine open V , we have

$$[V] = [S(U_1 \cap V, \dots, U_n \cap V)]$$

in $\mathbf{Gr}(X^{\text{pp}})$ by the first part of the proof. Hence, taking classes of both sides of (19) shows that

$$[Y]_{\mathcal{V}} = [S(U_1, \dots, U_n, V)] = [S_n(U_1, \dots, U_n)] = [Y]_{\mathcal{U}}$$

in $\mathbf{Gr}(X^{\text{pp}})$.

So, for Y an arbitrary X -scheme, we define $[Y] := [Y]_{\mathcal{U}}$, where \mathcal{U} is any finite open affine covering of Y . In particular, if Y is affine, we can take for open cover the singleton $\{Y\}$, showing that this new notation coincides with our former. To show that this assignment only depends on the isomorphism class of Y , let $\sigma: Y \rightarrow Y'$ be an isomorphism of X -schemes. Let \mathcal{U}' consist of all $\sigma(U)$ with $U \in \mathcal{U}$. Hence \mathcal{U}' is an open covering of Y' . Moreover, any intersection of members of \mathcal{U} is isomorphic to the intersection of the corresponding images under σ , and hence both have the same class in $\mathbf{Gr}(X^{\text{pp}})$. Therefore,

$$[Y] = [Y]_{\mathcal{U}} = [Y']_{\mathcal{U}'} = [Y'],$$

showing that the class of Y only depends on its isomorphism type.

Finally, to prove the last assertion, we first show that the additive map $[\cdot]: \mathbb{Z}[\Theta_Y] \rightarrow \mathbf{Gr}(X^{\text{pp}})$ factors through an additive map $\mathbf{Sciss}(\Theta_Y) \rightarrow \mathbf{Gr}(X^{\text{pp}}): U \mapsto [U]$, for every X -scheme Y (recall that Θ_Y is the lattice of all opens of Y). It suffices to show that any second scissor relation $U \cup U' = U + U' + U \cap U'$, with $U, U' \in \Theta_Y$, lies in the kernel of $\mathbb{Z}[\Theta_Y] \rightarrow \mathbf{Gr}(X^{\text{pp}})$. Let $\mathbf{U} := (U_1, \dots, U_n)$ and $\mathbf{U}' := (U'_1, \dots, U'_{n'})$ be affine open coverings of U and U' respectively. Hence the union of these two coverings is a covering of $U \cup U'$, whereas the collection \mathbf{V} of all $U_i \cap U'_j$, for $i = 1, \dots, n$, and $j = 1, \dots, n'$, is an affine covering of $U \cap U'$. Therefore, by (18), we have

$$(20) \quad [U \cup U'] - [U] - [U'] + [U \cap U'] = [S(\mathbf{U}, \mathbf{U}')] - [S(\mathbf{U})] - [S(\mathbf{U}')] + [S(\mathbf{V})]$$

However, by Lemma 2.1(4), we have an identity

$$S(\mathbf{U}, \mathbf{U}') - S(\mathbf{U}) - S(\mathbf{U}') + S(\mathbf{V}) = 0$$

in $\mathbb{Z}[\Theta_Y]$, showing that the right hand side of (20), whence also the left hand side, is zero. We can now prove the last assertion: let $\{U_1, \dots, U_n\}$ be an arbitrary finite open covering of Y . By Proposition 2.3, we have an identity

$$Y = \bigcup_{i=1}^n U_i = S(U_1, \dots, U_n)$$

in $\mathbf{Sciss}(\Theta_Y)$. Applying the additive map $\mathbf{Sciss}(\Theta_Y) \rightarrow \mathbf{Gr}(X^{\text{pp}})$ then yields the desired identity. \square

In the course of the proof, we obtained:

6.5. Corollary. *For each X -scheme Y , we have a homomorphism $\mathbf{K}_{\cong_X}(\Theta_Y) \rightarrow \mathbf{Gr}(X^{\text{pp}})$ of Grothendieck rings, where Θ_Y is the lattice of opens of Y , and \cong_X denotes isomorphism as X -schemes.* \square

6.6. Corollary. *If U is an open in an affine X -scheme Y , then there exists a disjunction ψ of schemic formulae such that $[U] = [\psi]$ in $\mathbf{Gr}(X^{\text{pp}})$.*

Proof. Let $\{D_1, \dots, D_n\}$ be a covering of U by basic open subsets, and, as in the proof of Proposition 6.2, let ψ_i be the schemic formula defining the basic open D_i , that is to say, $\psi_i(x, z) := \langle \phi(x) \wedge (f_i(x)z = 1) \rangle$, where ϕ is the defining formula of Y , and D_i is the basic open $\text{Spec}(\mathcal{O}_{Y, f_i})$. By Proposition 6.2, the basic open D_i is also defined by the pp-formula $\text{Im}(\psi_i)$. Let $\tilde{\psi} := \text{Im}(\psi_1) \vee \dots \vee \text{Im}(\psi_n)$. By Proposition 2.3, we have identities $U = S(D_1, \dots, D_n)$ in $\mathbf{Sciss}(\Theta_Y)$, and $\tilde{\psi} = S(\text{Im}(\psi_1), \dots, \text{Im}(\psi_n))$ in $\mathbf{Sciss}(\mathcal{PP})$. By Corollaries 6.5 and 3.5 respectively, we get $[U] = [S(D_1, \dots, D_n)]$ and $[\tilde{\psi}] = [S(\text{Im}(\psi_1), \dots, \text{Im}(\psi_n))]$ in $\mathbf{Gr}(X^{\text{pp}})$. One easily verifies that any conjunction $\bigwedge_{i \in I} \text{Im}(\psi_i)$ is the defining pp-formula for the corresponding intersection of the D_i , for $I \subseteq \{1, \dots, n\}$ (but see Remark 6.3 for why we cannot work with the schemic formulae ψ_i instead). Hence

$$[U] = [S(D_1, \dots, D_n)] = [S(\text{Im}(\psi_1), \dots, \text{Im}(\psi_n))] = [\tilde{\psi}]$$

in $\mathbf{Gr}(X^{\text{pp}})$.

To obtain a disjunction of schemic formulae, let

$$\psi(x, z_1, \dots, z_n) := \bigvee_{i=1}^n \psi_i(x, z_i).$$

As in the proof of Proposition 6.2, one can show that the projection onto the x -coordinates yields a (schemic) isomorphism between ψ and $\tilde{\psi}$ modulo \mathbf{Art}_X , and hence $[\psi] = [\tilde{\psi}]$ in $\mathbf{Gr}(X^{\text{pp}})$, completing the proof of the assertion. \square

6.7. Remark. Theorem 6.4 allows us to calculate the class of a non-affine scheme in terms of classes of affine schemes. For instance, the class of the projective line \mathbb{P}_X^1 is equal to $2\mathbb{L} - \mathbb{L}^*$, where, as before, \mathbb{L} is the Lefschetz class, that is to say, the class of the affine line \mathbb{A}_X^1 , and where \mathbb{L}^* denotes the class of the affine line without the origin. One would be tempted to think that $\mathbb{L}^* = \mathbb{L} - 1$, but this is false, for the reason given at end of Remark 6.3. We will give a correct version of this formula in (33) below.

7. ARC INTEGRALS

Let $X = \text{Spec } A$ be an affine Noetherian scheme, and let ϕ be a schemic formula in \mathcal{L}_A with corresponding ideal $I(\phi)$, and associated affine scheme $Y_\phi := \text{Spec}(A[x]/I(\phi))$. We call ϕ *Artinian* if the corresponding affine scheme Y_ϕ is Artinian, that is to say, has (Krull) dimension zero. We denote the Boolean closure of the collection of Artinian formulae by \mathcal{Art} (not to be confused with the theory \mathbf{Art}_A). We say that ϕ is a *closed point formula*, if $I(\phi)$ is a maximal ideal; we say that ϕ is a *point formula*, if the radical of $I(\phi)$ is a maximal ideal. Closed point formulae and point formulae are Artinian. Let ϕ be a point formula. There is a unique closed point formula $\bar{\phi}$ implying ϕ , namely, the one corresponding to the radical of $I(\phi)$. To a point formula corresponds an Artinian local A -scheme Y_ϕ , and the closed point of Y_ϕ then corresponds to $\bar{\phi}$. If $A = F$ is an algebraically closed field, then any two point formulae are \mathbf{T}_F -equivalent by the Nullstellensatz, but this might fail in general. For instance, over $A = \mathbb{Q}$, the formulae $x^2 + 1 = 0$ and $x = 0$ are not isomorphic. Another example, with $A = F[[t]]$, are the point formulae $tx - 1 = 0$ and $t = x = 0$, which cannot be isomorphic, not even after a base change. Let us denote the Grothendieck ring of Artinian formulae modulo \mathbf{Art}_X by

$$\mathbf{Gr}_0(X^{\text{sch}}) := \mathbf{Gr}_{\text{Sch}}^{\mathbf{Art}_X}(\mathcal{Art}).$$

Given an Artinian X -scheme Y , we will call its *length* $\ell(Y)$ the length of the coordinate ring \mathcal{O}_Y viewed as an A -module.

7.1. Corollary. *Any element of $\mathbf{Gr}_0(X^{\text{sch}})$ can be written as a difference $[Y] - [Y']$, with Y and Y' Artinian X -schemes.*

Proof. By Lemma 4.2, Artinian formulae are closed under disjoint sums, and the claim follows from Corollary 3.9. \square

7.2. Lemma. *Any Artinian formula is a disjoint union of finitely many point formulae.*

Proof. Immediate from the fact that an Artinian ring is a direct sum of Artinian local rings. \square

For simplicity, we will for the remainder of this section work over an algebraically closed field F .

7.3. Lemma. *Let F be an algebraically closed field. There exists a ring homomorphism $\ell: \mathbf{Gr}_0(F^{\text{sch}}) \rightarrow \mathbb{Z}$ such that $\ell([Y]) = \ell(Y)$ for any Artinian F -scheme Y .*

Proof. Since F is algebraically closed, $\ell(Y)$ is equal to the F -vector space dimension of \mathcal{O}_Y . Let ϕ be a formula in $\mathcal{A}rt$. By Corollary 7.1, we can find Artinian F -schemes Y and Y' such that $[\phi] = [Y] - [Y']$. Put $\ell([\phi]) := \ell(Y) - \ell(Y')$. To prove that this is independent from the choice of affine schemes, suppose that also $[\phi] = [Z] - [Z']$ for some Artinian F -schemes Z and Z' . It follows that $[Y \sqcup Z'] = [Y' \sqcup Z]$, and hence by Theorem 3.11, that the schemes $Y \sqcup Z'$ and $Y' \sqcup Z$ are stably $\mathcal{S}ch$ -isomorphic in $\mathcal{A}rt$. This means that $Y \sqcup Z' \sqcup T$ and $Y' \sqcup Z \sqcup T$ are isomorphic, for some Artinian F -scheme T . Since length is additive on disjoint unions, $\ell(Y) + \ell(Z') + \ell(T) = \ell(Y') + \ell(Z) + \ell(T)$, showing that $\ell([\phi])$ is well-defined. Linearity follows immediately from this. Furthermore, given two Artinian F -schemes, we have $\ell([Y] \cdot [Z]) = \ell([Y \times_F Z])$. Let l and m be the length of Y and Z respectively, so that as F -vector spaces $\mathcal{O}_Y \cong F^l$ and $\mathcal{O}_Z \cong F^m$. Since the coordinate ring of $Y \times_F Z$ is equal to $\mathcal{O}_Y \otimes_F \mathcal{O}_Z$, its length is equal to mn , showing that ℓ is also multiplicative. \square

Jets. Given a closed subscheme Z of an affine scheme $Y := \text{Spec } B$, we define the n -th jet of Y along Z to be the closed subscheme

$$J_Z^n Y := \text{Spec}(B/I^n),$$

where I is the ideal of the closed subscheme Z . Note that Z and any of the jets $J_Z^n Y$ have the same underlying topological space, and we have an ascending chain of closed subschemes

$$(21) \quad \emptyset = J_Z^0 Y \subseteq Z = J_Z^1 Y \subseteq J_Z^2 Y \cdots \subseteq J_Z^n Y \subseteq \dots$$

In most cases, this will be a proper chain by Nakayama's Lemma (for instance, if Z is a proper closed subscheme of a variety Y , or more generally, if I contains a non-zero divisor).

We can generalize the notion of a jet to formulae: let ϕ be an arbitrary formula and ζ a schemic formula. We define the n -th jet of ϕ along ζ to be the formula

$$J_\zeta^n \phi := \phi \wedge \bigwedge_{f_1, \dots, f_n \in I(\zeta)} (f_1 \cdot f_2 \cdots f_n = 0).$$

In other words, if $\zeta^{(n)}$ is the formula with defining ideal $I(\zeta)^n$, then $J_\zeta^n \phi = \phi \wedge \zeta^{(n)}$. In particular, if ϕ is also schemic, defining an affine variety $Y := Y_\phi$, and if Z is the closed subscheme defined by $\phi \wedge \zeta$, then $J_\zeta^n \phi$ is the defining schemic formula of $J_Z^n Y$.

Formal Hilbert series. Let ϕ be a schemic formula, and τ a closed point formula implying ϕ . Each jet $J_\tau^i \phi$ is an Artinian formula, and hence its class belongs to $\mathbf{Gr}_0(F^{\text{sch}})$. In particular, if X is the F -scheme defined by ϕ , and P the closed point defined by τ , then $J_\tau^i \phi = J_P^i X$. For T a single variable we can therefore define the *formal Hilbert series* of a F -scheme X at a closed point P as the series

$$\text{Hilb}_P(X) := \sum_{i=0}^{\infty} [J_P^i X] T^i$$

in $\mathbf{Gr}_0(F^{\text{sch}})[[T]]$. If we extend the homomorphism ℓ to $\mathbf{Gr}_0(F^{\text{sch}})[[T]]$ by letting it act on the coefficients of a power series, then $\ell(\text{Hilb}_P(X))$ is a rational function in $\mathbb{Z}[[T]]$ by the Hilbert-Samuel theory (it is the first difference of the classical Hilbert series of X at P).

Arcs. Let R be an Artinian algebra of dimension l over F , and fix some basis Δ of R over F . For each $\alpha \in \Delta$, we define the α -th coordinate map $\pi_\alpha: R \rightarrow F$ by the rule

$$r = \sum_{\alpha \in \Delta} \pi_\alpha(r) \cdot \alpha.$$

We write $\pi_R(r)$, or just $\pi(r)$, for the tuple of all $\pi_\alpha(r)$, where we fix once and for all an order of Δ . In particular, π gives a (F -linear) bijection between R and F^l . We also extend this notation to arbitrary tuples. More generally, if A is a F -algebra, then $R \otimes_F A$ is a free A -module generated by Δ and the base change of π yields an A -linear isomorphism $R \otimes_F A \cong A^l$, which we continue to denote by π . In this section, we also will fix the following notation. Given an n -tuple of variables x , we let \tilde{x}_α , for each $\alpha \in \Delta$, be another n -tuple of variables, and we denote the ln -tuple consisting of all \tilde{x}_α by \tilde{x} , referring to them as *arc variables*. We also associate to each n -tuple of variables x , an n -tuple of *generic arcs*

$$\dot{x} := \sum_{\alpha \in \Delta} \tilde{x}_\alpha \alpha$$

viewed as a tuple in $R[\tilde{x}]$. In particular, $\pi_\alpha(\dot{x}) = \tilde{x}_\alpha$.

7.4. Proposition. *For each \mathcal{L}_F -formula ϕ of arity n , and for each finite F -algebra R of dimension l , there exists an \mathcal{L}_F -formula $\nabla_R \phi$ of arity ln with the following property: if A is an F -algebra and a an n -tuple in $R \otimes_F A$, then $a \in \phi(R \otimes_F A)$ if and only if $\pi(a) \in \nabla_R \phi(A)$. Moreover, if ϕ is schemic or pp, then so is $\nabla_R \phi$.*

Proof. Let Δ be a basis of R as a vector space over F . For each polynomial $f \in F[x]$, define polynomials $\nabla_\alpha f \in F[\tilde{x}]$ by the rule

$$(22) \quad f(\dot{x}) = f\left(\sum_{\alpha} \tilde{x}_\alpha \alpha\right) = \sum_{\alpha \in \Delta} \nabla_\alpha f(\tilde{x}) \alpha.$$

In particular, for A an F -algebra, we have $\pi_\alpha(f(a)) = (\nabla_\alpha f)(\pi(a))$, for all a in $R \otimes_F A$ and all $\alpha \in \Delta$. To define $\nabla_R \phi(\tilde{x})$, we induct on the complexity of the formula ϕ . If ϕ is the schemic formula $f(x) = 0$, then $\nabla_R \phi$ is the schemic formula

$$\nabla_R \phi(\tilde{x}) := \langle \bigwedge_{\alpha \in \Delta} (\nabla_\alpha f(\tilde{x}) = 0) \rangle.$$

If ϕ and ψ are formulae for which we already defined $\nabla_R \phi$ and $\nabla_R \psi$, then $\nabla_R(\phi \vee \psi) := \nabla_R \phi \vee \nabla_R \psi$, and $\nabla_R(\neg \phi) := \neg \nabla_R \phi$. Finally, if $\phi(x)$ is the formula $(\exists y)\psi(x, y)$, then we define $\nabla_R \phi$ as the formula

$$\nabla_R \phi(x) := \langle (\exists \tilde{y}) \nabla_R \psi(\tilde{x}, \tilde{y}) \rangle$$

where, similarly, \tilde{y} is a tuple of l copies of y . This concludes the proof of the existence of $\nabla_R\phi$. That it satisfies the desired one-one correspondence between definable sets is clear from (22), and the last assertion is immediate as well. \square

We will refer to $\nabla_R\phi$ as the *arc formula of ϕ along R* . Instead of using R as a subscript, we may also use its defining schemic formula, or the Artinian scheme $Z := \text{Spec } R$ it determines, or even leave out reference to it altogether, whenever it is clear from the context. If ϕ is a schemic formula defining an affine scheme Y , then we will write $\nabla_Z Y$ for the affine scheme determined by $\nabla_R\phi$, and call it the *arc scheme of Y along Z* . The following shows that arcs along an Artinian scheme are generalizations of truncated arcs.

7.5. Proposition. *Let Z be an Artinian F -scheme, and let X and Y be affine F -schemes. There is a one-one correspondence between $Z \times_F X$ -rational points on $Y \times_F X$ over X , and X -rational points on the corresponding arc scheme $\nabla_Z Y$ over F , that is to say, we have a one-one correspondence*

$$\text{Mor}_X(Z \times_F X, Y \times_F X) \cong \text{Mor}_F(X, \nabla_Z Y).$$

Proof. Let ϕ be the schemic formula defining Y , and let R and X be the respective coordinate rings of Z and X . Viewing ϕ in the language \mathcal{L}_X , it is the schemic formula defining the affine X -scheme $Y \times X$, by §4.1.1. By Lemma 4.5, we may identify $\phi(R \otimes_F A)$ with $\text{Mor}_F(Z \times X, Y \times X)$, and similarly, $\nabla_R\phi(A)$ with $\text{Mor}_F(X, \nabla_Z Y)$. The result then follows from Proposition 7.4. \square

In particular, $\text{Mor}_F(Z, Y) = \nabla_Z Y(F)$. For instance, if $Z_n := F[\xi]/\xi^n F[\xi]$, then $(\nabla_{Z_n} Y)^{\text{red}}$ is the truncated arc space $\mathcal{L}_n(Y)$ as defined in [1, p. 276] or [6].

7.6. Remark. Given a morphism of schemes $Z \rightarrow Y$ over an arbitrary base scheme S , we may view $\text{Mor}_S(Z, Y)$ as a contravariant functor on the category of S -schemes through base change, that is to say,

$$\text{Mor}_S(Z, Y)(X) := \text{Mor}_X(Z \times_S X, Y \times_S X),$$

for any S -scheme X . The content of Proposition 7.5 is then that for any Artinian scheme Z over an algebraically closed field F , and any affine F -scheme Y , the functor $\text{Mor}_F(Z, Y)$ is representable. Indeed, in the definition of representability, it suffices to consider only affine F -schemes X , since $\text{Mor}_F(Z, Y)$ is compatible with limits, yielding that the affine F -scheme $\nabla_Z Y$ represents $\text{Mor}_F(Z, Y)$.

7.7. Example. Before we proceed, some simple examples are in order. It is clear from the definitions that $\nabla_R\lambda = \lambda_l \cong \lambda^l$, for λ the Lefschetz formula $x = x$.

Let us next calculate the arc scheme of the curve given by the formula $\phi := \langle x^2 = y^3 \rangle$ along the four dimensional algebra $R := F[\xi, \zeta]/(\xi^2, \zeta^2)F[\xi, \zeta]$, using the basis $\Delta := \{1, \xi, \zeta, \xi\zeta\}$ (in the order listed), and corresponding arc variables the quadruples $\tilde{x} = (\tilde{x}_{(0,0)}, \tilde{x}_{(1,0)}, \tilde{x}_{(0,1)}, \tilde{x}_{(1,1)})$ and $\tilde{y} = (\tilde{y}_{(0,0)}, \tilde{y}_{(1,0)}, \tilde{y}_{(0,1)}, \tilde{y}_{(1,1)})$. One easily calculates that $\nabla_R\phi$ is the schemic formula

$$\begin{aligned} \tilde{x}_{(0,0)}^2 &= \tilde{y}_{(0,0)}^3 \\ 2\tilde{x}_{(0,0)}\tilde{x}_{(1,0)} &= 3\tilde{y}_{(0,0)}^2\tilde{y}_{(1,0)} \\ 2\tilde{x}_{(0,0)}\tilde{x}_{(0,1)} &= 3\tilde{y}_{(0,0)}^2\tilde{y}_{(0,1)} \\ 2\tilde{x}_{(0,0)}\tilde{x}_{(1,1)} + 2\tilde{x}_{(1,0)}\tilde{x}_{(0,1)} &= 3\tilde{y}_{(0,0)}^2\tilde{y}_{(1,1)} + 6\tilde{y}_{(0,0)}\tilde{y}_{(1,0)}\tilde{y}_{(0,1)}. \end{aligned}$$

Note that the first equation is $\phi(\tilde{x}_{(0,0)}, \tilde{y}_{(0,0)})$, and that above the singular point $\tilde{x}_{(0,0)} = 0 = \tilde{y}_{(0,0)}$, the fiber consist of two 4-dimensional planes.

7.8. Example. Another example is classical: let $R = F[\xi]/\xi^2 F[\xi]$ be the ring of dual numbers. Then one verifies that a F -rational point on $\nabla_R Y$ is given by a F -rational point P on Y , and a tangent vector \mathbf{v} to Y at P , that is to say, an element in the kernel of the Jacobian matrix $\text{Jac}_\phi(P)$.

7.9. Example. As a last example, we calculate $\nabla_{Z_n} Z_m$, where $Z_n := \text{Spec}(F[\xi]/\xi^n F[\xi])$. With $\dot{x} = \tilde{x}_0 + \tilde{x}_1 \xi + \cdots + \tilde{x}_{n-1} \xi^{n-1}$, we will expand \dot{x}^m in the basis $\{1, \xi, \dots, \xi^{n-1}\}$ of $F[\xi]/\xi^n F[\xi]$ (see Lemma 7.10 below for why the choice of basis is not important); the coefficients of this expansion then generate the ideal of definition of $\nabla_{Z_n} Z_m$. A quick calculation shows that these generators are the polynomials

$$g_s(\tilde{x}_0, \dots, \tilde{x}_{n-1}) := \sum_{i_1 + \cdots + i_m = s} \tilde{x}_{i_1} \tilde{x}_{i_2} \cdots \tilde{x}_{i_m}$$

for $s = 0, \dots, n-1$, where the i_j run over $\{0, \dots, n-1\}$. Note that $g_0 = \tilde{x}_0^m$. One shows by induction that $(\tilde{x}_0, \dots, \tilde{x}_s) F[\tilde{x}]$ is the unique minimal prime ideal of $\nabla_{Z_n} Z_m$, where $s = \lceil \frac{n}{m} \rceil$ is the *round-up* of n/m , that is to say, the least integer greater than or equal to n/m . In particular, $\nabla_{Z_n} Z_m$ is irreducible of dimension $n - \lceil \frac{n}{m} \rceil$.

Although the arc scheme depends on the choice of basis, we have:

7.10. Lemma. *For each finite dimensional F -algebra R , and each \mathcal{L}_F -formula ϕ , the arc formula $\nabla_R \phi$ along R is unique up to an explicit isomorphism modulo \mathbf{Art}_F .*

Proof. Suppose R has dimension l over F , and let Δ and Δ^* be two bases of R , with corresponding isomorphisms π and π_* between R and F^l , and corresponding arc maps ∇ and ∇_* on \mathcal{L}_F . There exists an F -linear automorphism σ of R sending Δ to Δ^* . Applying σ to $r = \sum \pi_\alpha(r) \alpha$ yields $\sigma(r) = \sum \pi_\alpha(r) \sigma(\alpha)$, showing that

$$(23) \quad \pi_*(\sigma(r)) = \pi(r),$$

for any $r \in R$. Define τ as the automorphism $\pi \circ \sigma^{-1} \circ \pi^{-1}$ of F^l . I claim that the explicit formula $\tilde{y} = \tau(\tilde{x})$ induces an isomorphism between $\nabla \phi(\tilde{x})$ and $\nabla_* \phi(\tilde{y})$ modulo \mathbf{T}_F (whence also modulo \mathbf{Art}_F), for any formula ϕ . Indeed, let A be a finitely generated F -algebra, and let $u \in \nabla \phi(A)$. Put $a := \pi^{-1} u$, where we continue to write π for the base change $A' := R \otimes_F A \rightarrow A^l$. Applying Proposition 7.4 twice, we get $a \in \phi(A')$ whence $\pi_*(a) \in \nabla_* \phi(A)$. Since $\tau(u) = \pi(\sigma^{-1}(a)) = \pi_*(a)$ by a component-wise application of (23), we showed that τ induces the desired isomorphism between $\nabla \phi(A)$ and $\nabla_* \phi(A)$. \square

7.11. Remark. So, from now on, we may choose a basis $\Delta = \{\alpha_0, \dots, \alpha_{l-1}\}$ of (R, \mathfrak{m}) with some additional properties. In particular, unless noted explicitly, we will always assume that the first base element is 1 and that the remaining ones belong to \mathfrak{m} . Moreover, once the basis is fixed, we let \tilde{x} be the l -tuple of arc variables $(\tilde{x}_0, \dots, \tilde{x}_{l-1})$, so that $\dot{x} = \tilde{x}_0 + \tilde{x}_1 \alpha_1 + \cdots + \tilde{x}_{l-1} \alpha_{l-1}$ is the corresponding generic arc. It follows from (22) that $\nabla_0 f = f(\tilde{x}_0)$, for any $f \in F[x]$, where henceforth we simply write $\nabla_j f$ for $\nabla_{\alpha_j} f$. By [15, §2.1], we may choose Δ so that, with $\mathfrak{a}_i := (\alpha_i, \dots, \alpha_{l-1})R$, we have a Jordan-Holder composition series⁴

$$\mathfrak{a}_l = 0 \subsetneq \mathfrak{a}_{l-1} \subsetneq \mathfrak{a}_{l-2} \subsetneq \cdots \subsetneq \mathfrak{a}_1 = \mathfrak{m} \subsetneq \mathfrak{a}_0 = R.$$

I claim that π_j vanishes on each element in \mathfrak{a}_j for $j < i$. Indeed, if not, let $j < i$ be minimal so that there exists a counterexample with $r_j := \pi_j(r) \neq 0$ for some $r \in \mathfrak{a}_i$. By

⁴Writing R as a homomorphic image of $F[y]$ so that $y := (y_1, \dots, y_e)$ generates \mathfrak{m} , let $\mathfrak{a}(\alpha)$, for $\alpha \in \mathbb{Z}_{\geq 0}^n$, be the ideal in R generated by all y^β with β lexicographically larger than α . Then we may take Δ to be all monomials y^α such that $y^\alpha \notin \mathfrak{a}(\alpha)$, ordered lexicographically.

minimality, $r = r_j \alpha_j + r_{j+1} \alpha_{j+1} + \dots \in \mathfrak{a}_i$ showing that $\alpha_j \in \mathfrak{a}_{j+1}$, since r_j is invertible. However, this implies that $\mathfrak{a}_j = \mathfrak{a}_{j+1}$, contradiction.

From this, it is now easy to see that the first m basis elements of Δ form a basis of $R_m := R/\mathfrak{a}_{m+1}$. Put differently, if $r \in R$, then the m -tuple $\pi_{R_m}(r)$ is the initial part of the l -tuple $\pi_R(r)$. Therefore, calculating $\nabla_m f$ for $f \in F[x]$ does not depend on whether we work with π_R or with π_{R_m} , and hence, in particular, $\nabla_m f \in F[\tilde{x}_0, \dots, \tilde{x}_m]$ for every $m < l$.

The next result together with Corollary 7.20 below shows that arc schemes are functorial fibrations:

7.12. Theorem. *Let Z be a local Artinian F -scheme of length l . For each affine F -scheme $X \subseteq \mathbb{A}_F^n$, the projection $\mathbb{A}_F^{ln} \rightarrow \mathbb{A}_F^n$ onto the first n coordinates induces a split surjective map $\nabla_Z X \rightarrow X$, which is smooth above the regular locus of X . If $h: Y \rightarrow X$ is a morphism of affine F -schemes, then we have an induced morphism $\nabla_Z h: \nabla_Z Y \rightarrow \nabla_Z X$ making the diagram*

$$(24) \quad \begin{array}{ccc} \nabla_Z Y & \xrightarrow{\nabla_Z h} & \nabla_Z X \\ \downarrow & & \downarrow \\ Y & \xrightarrow{h} & X \end{array}$$

commute. Moreover, if h is a closed immersion, then so is $\nabla_Z h$. If $Y \subseteq X$ is an open immersion, then we even have an isomorphism

$$(25) \quad \nabla_Z Y \cong \nabla_Z X \times_X Y.$$

Proof. Let (R, \mathfrak{m}) be the Artinian local ring \mathcal{O}_Z corresponding to Z , and calculate $\pi := \pi_R$ with the basis given as in Remark 7.11. The projection $\mathbb{A}_F^{ln} \rightarrow \mathbb{A}_F^n$ is given by the embedding $F[x] \rightarrow F[\tilde{x}]: x \mapsto \tilde{x}_0$. Let ϕ be the schemic formula defining X , let $I := I(\phi)$ be the corresponding ideal, and let $A := F[x]/I$ be its coordinate ring. Furthermore, let $\tilde{I} := I(\nabla X)$ be the ideal defining $\nabla X := \nabla_Z X$, and let $\tilde{A} := F[\tilde{x}]/\tilde{I}$ be its coordinate ring. The existence of the map $\nabla X \rightarrow X$ follows from our observation in Remark 7.11 that $\nabla_0 f = f(\tilde{x}_0)$. Namely, applying this to every equation in ϕ , we see that the explicit formula $\tilde{x}_0 = x$ defining the projection $\mathbb{A}_F^{ln} \rightarrow \mathbb{A}_F^n$ induces a morphism $\phi \rightarrow \nabla \phi$, whence a homomorphism $A \rightarrow \tilde{A}$, that is to say, a morphism $\nabla_Z X \rightarrow X$. Let \mathfrak{b} be the ideal in $F[\tilde{x}]$ generated by all \tilde{x}_i with $0 < i < l$. Since $\tilde{x} \equiv \tilde{x}_0 \pmod{\mathfrak{b}R[\tilde{x}]}$, equation (22) yields that all $\nabla_u f$ belong to \mathfrak{b} , for $u > 0$. Hence $\tilde{A}/\mathfrak{b}\tilde{A} \cong A$, showing that $\nabla X \rightarrow X$ has a section, whence is split surjective.

If $Y \rightarrow X$ is a morphism of affine F -schemes, then this corresponds by Corollary 4.8 to an explicit morphism $\phi \rightarrow \psi$, where ψ is the schemic formula defining Y . We leave it to the reader to verify that this induces an explicit morphism $\nabla \phi \rightarrow \nabla \psi$, leading to a commutative diagram (24). Suppose $Y \rightarrow X$ is a closed immersion. We may assume that Y is a closed subscheme of X , and hence the schemic formula of Y can be taken to be a conjunction of the form $\phi \wedge \psi$, with ψ some schemic formula. Therefore, $\nabla(\phi \wedge \psi) = \nabla \phi \wedge \nabla \psi$ is the schemic formula for ∇Y , showing that it is a closed subscheme of ∇X . Next suppose $Y \rightarrow X$ is an open immersion. We may reduce to the case that Y is a basic open subset of X , since ∇ is compatible with disjuncts/unions. By the case of a

closed immersion just proved and the fact that ∇ also preserves intersections, we may furthermore reduce to the case that $Y = D(f) \subseteq X = \mathbb{A}_F^n$ for some non-zero $f \in F[x]$. Hence Y is defined, as a closed subscheme of \mathbb{A}_F^{n+1} , by $g(x, y) := yf(x) - 1$. Let $\tilde{y} = (\tilde{y}_0, \dots, \tilde{y}_{l-1})$ be arc variables with corresponding generic arc $\dot{y} := \tilde{y}_0 + \dots + \tilde{y}_{l-1}\alpha_{l-1}$. Let $\tilde{J} \subseteq F[\tilde{x}, \tilde{y}]$ be the ideal defining ∇Y , that is to say, the ideal generated by all $\nabla_u g$, and let $\tilde{B} := F[\tilde{x}, \tilde{y}]/\tilde{J}$ be the coordinate ring of ∇Y . Our aim is to show that ∇Y is isomorphic to $D(\nabla_0 f) \subseteq \mathbb{A}_F^{ln}$. Using (22), we get

$$\sum_{u=0}^{l-1} \nabla_u g(\tilde{x}, \tilde{y})\alpha_u = g(\dot{x}, \dot{y}) = \dot{y} \cdot \left(\sum_{u=0}^{l-1} \nabla_u f(\tilde{x})\alpha_u \right) - 1.$$

Expansion yields $\nabla_0 f = f(\tilde{x}_0)$ and $\nabla_0 g = \tilde{y}_0 \nabla_0 f - 1$. Hence under the canonical homomorphism $F[x] \rightarrow \tilde{B}$ given by $x \mapsto \tilde{x}_0$, we get $\tilde{y}_0 f = 1$ in \tilde{B} . By Remark 7.11, in order to calculate $\nabla_u g$ for $u > 0$, we may ignore all terms containing some α_i with $i > u$, that is to say,

$$\begin{aligned} \nabla_u g &= \nabla_u \left((\tilde{y}_0 + \dots + \tilde{y}_u \alpha_u) (\nabla_0 f + \dots + (\nabla_u f)\alpha_u) \right) \\ &= \tilde{y}_u \nabla_0 f + \nabla_u \left((\tilde{y}_0 + \dots + \tilde{y}_{u-1} \alpha_{u-1}) (\nabla_0 f + \dots + (\nabla_u f)\alpha_u) \right). \end{aligned}$$

Note that the second term lies in $F[\tilde{x}, \tilde{y}_0, \dots, \tilde{y}_{u-1}]$. Since $\nabla_0 f = f$ and since $y_0 f = 1$ and $\nabla_u g = 0$ in \tilde{B} , we obtain, after multiplying this sum by \tilde{y}_0 , that \tilde{y}_u lies in the F -subalgebra of \tilde{B} generated by all \tilde{x} and all \tilde{y}_j with $j < u$. Hence, by downward induction on u , we get

$$\tilde{B} \cong F[\tilde{x}, \tilde{y}_0]/(\tilde{y}_0 f - 1)F[\tilde{x}, \tilde{y}_0],$$

showing that $\nabla Y = D(f)$, as claimed. In particular, ∇Y is the pull-back of Y under the map $\nabla X \rightarrow X$, that is to say, (25) holds.

So remains to show that if P is a closed point in the regular locus of X , then the fiber of $\nabla X \rightarrow X$ at P is non-singular. By the Nullstellensatz, we may, after a change of variables (translation), assume that P corresponds to the maximal ideal $\mathfrak{n} := (x_1, \dots, x_n)F[x]$. Since $\mathcal{O}_{X,P} = A_{\mathfrak{n},A}$ is a regular local ring, $IF[x]_{\mathfrak{n}}$ is generated by a regular system of parameters ([14, Theorem 14.2]), say, of length h . Hence, by Nakayama's Lemma, we can find an open $U \subseteq \mathbb{A}_F^n$ containing P , such that IO_U is generated by h elements whose image in $\mathcal{O}_{U,P}$ are part of a generating system of \mathfrak{n} . Since $\nabla(U \cap X)$ is just the pull-back of $U \cap X$ by (25), and since the present question is not affected by such a pull-back, we may take $X = U$, and assume that $I = (f_1, \dots, f_h)F[x]$, with the f_u part of a minimal system of generators of \mathfrak{n} . In particular, the linear parts of the f_u must be linearly independent over F . Hence after a linear change of variables (rotation), we may assume that $f_u = x_u + g_u$, with each $g_u \in \mathfrak{n}^2$, for $j = 1, \dots, h$. Using (22), we get

$$f_v(\dot{x}) = \tilde{x}_{0,v} + \tilde{x}_{1,v}\alpha_1 + \dots + \tilde{x}_{l-1,v}\alpha_{l-1} + \sum_{u=0}^{l-1} \nabla_u g_v(\tilde{x})\alpha_u$$

for $v = 1, \dots, h$, showing that $\nabla\phi$ is the conjunction of the lh equations $\nabla_u f_v = \tilde{x}_{u,v} + \nabla_u g_v = 0$, with $u = 0, \dots, l-1$ and $v = 1, \dots, h$.

Let $J(\tilde{x})$ be the $lh \times lh$ -submatrix

$$\begin{pmatrix} \frac{\partial(\tilde{x}_{0,1} + \nabla_0 g_1)}{\partial \tilde{x}_{0,1}} & \cdots & \frac{\partial(\tilde{x}_{0,1} + \nabla_0 g_1)}{\partial \tilde{x}_{l-1,1}} & \frac{\partial(\tilde{x}_{0,1} + \nabla_0 g_1)}{\partial \tilde{x}_{0,2}} & \cdots & \frac{\partial(\tilde{x}_{0,1} + \nabla_0 g_1)}{\partial \tilde{x}_{l-1,h}} \\ \frac{\partial(\tilde{x}_{1,1} + \nabla_1 g_1)}{\partial \tilde{x}_{0,1}} & \cdots & \frac{\partial(\tilde{x}_{1,1} + \nabla_1 g_1)}{\partial \tilde{x}_{l-1,1}} & \frac{\partial(\tilde{x}_{1,1} + \nabla_1 g_1)}{\partial \tilde{x}_{0,2}} & \cdots & \frac{\partial(\tilde{x}_{1,1} + \nabla_1 g_1)}{\partial \tilde{x}_{l-1,h}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\tilde{x}_{l-1,1} + \nabla_{l-1} g_1)}{\partial \tilde{x}_{0,1}} & \cdots & \frac{\partial(\tilde{x}_{l-1,1} + \nabla_{l-1} g_1)}{\partial \tilde{x}_{l-1,1}} & \frac{\partial(\tilde{x}_{l-1,1} + \nabla_{l-1} g_1)}{\partial \tilde{x}_{0,2}} & \cdots & \frac{\partial(\tilde{x}_{l-1,1} + \nabla_{l-1} g_1)}{\partial \tilde{x}_{l-1,h}} \\ \frac{\partial(\tilde{x}_{0,2} + \nabla_0 g_2)}{\partial \tilde{x}_{0,1}} & \cdots & \frac{\partial(\tilde{x}_{0,2} + \nabla_0 g_2)}{\partial \tilde{x}_{l-1,1}} & \frac{\partial(\tilde{x}_{0,2} + \nabla_0 g_2)}{\partial \tilde{x}_{0,2}} & \cdots & \frac{\partial(\tilde{x}_{0,2} + \nabla_0 g_2)}{\partial \tilde{x}_{l-1,h}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial(\tilde{x}_{l-1,h} + \nabla_{l-1} g_h)}{\partial \tilde{x}_{0,1}} & \cdots & \frac{\partial(\tilde{x}_{l-1,h} + \nabla_{l-1} g_h)}{\partial \tilde{x}_{l-1,1}} & \frac{\partial(\tilde{x}_{l-1,h} + \nabla_{l-1} g_h)}{\partial \tilde{x}_{0,2}} & \cdots & \frac{\partial(\tilde{x}_{l-1,h} + \nabla_{l-1} g_h)}{\partial \tilde{x}_{l-1,h}} \end{pmatrix}$$

of the Jacobian $\text{Jac}_{\nabla\phi}(\tilde{x})$, and, for each $1 \leq v, j \leq h$, let $J_{v,j}$ be the $l \times l$ submatrix $(\partial(\tilde{x}_{u,v} + \nabla_u g_v)/\partial \tilde{x}_{i,j})_{u,i=0,\dots,l-1}$ of J . I claim that each $J_{v,j}$ is an upper-triangular matrix with diagonal entries all equal to $\partial f_v/\partial x_j$ (under the canonical embedding $F[x] \rightarrow F[\tilde{x}]$). Assuming the claim, let \tilde{P} be an arbitrary closed point in the fiber above P . In particular, if $\tilde{\mathfrak{n}} \subseteq F[\tilde{x}]$ is the maximal ideal corresponding to \tilde{P} , then $\mathfrak{n} = \tilde{\mathfrak{n}} \cap F[x]$. Hence, the determinant of $J_{v,j}$ evaluated at \tilde{P} is equal to $(\partial f_v/\partial x_j)^l(P)$, which is equal to the Kronecker delta function $\delta_{v,j}$. This shows that $J(\tilde{P})$ is an upper-triangular matrix with determinant equal to one, from which it follows that $\text{Jac}_{\nabla\phi}(\tilde{P})$ has rank at least lh . Since $I(\nabla\phi)$ has height at most lh by Krull's Principal Ideal theorem, we showed, using the Jacobian criterion for smoothness ([8, Theorem 16.19]), that ∇X is smooth at \tilde{P} , as we wanted to show.

So remains to prove the claim. For $f \in F[x]$, we apply the (i, j) -th partial derivative to both sides of (22). On the left hand side, we get, using the chain rule

$$\begin{aligned} \frac{\partial(f(\dot{x}))}{\partial \tilde{x}_{i,j}} &= \frac{\partial f}{\partial x_j}(\dot{x}) \cdot \frac{\partial(\dot{x}_j)}{\partial \tilde{x}_{i,j}} \\ &= \frac{\partial f}{\partial x_j}(\dot{x}) \cdot \alpha_i. \end{aligned}$$

Doing the same to the right hand side of (22), we get

$$(26) \quad \frac{\partial f}{\partial x_j}(\dot{x}) \cdot \alpha_i = \sum_{u=0}^{l-1} \frac{\partial(\nabla_u f)}{\partial \tilde{x}_{i,j}} \alpha_u.$$

In view of (22) for $\partial f/\partial x_j$, the left hand side of (26) becomes

$$\left(\frac{\partial f}{\partial x_j} + \nabla_1 \left(\frac{\partial f}{\partial x_j} \right) \alpha_1 + \cdots + \nabla_{l-1} \left(\frac{\partial f}{\partial x_j} \right) \alpha_{l-1} \right) \cdot \alpha_i.$$

By the choice of basis (which remains a basis for the base change $R[\tilde{x}]$ over $F[\tilde{x}]$), the coefficient of α_i in this product is $\partial f/\partial x_j$. Hence comparing this with the right hand side of (26), we obtain

$$\frac{\partial f}{\partial x_j} = \frac{\partial(\nabla_i f)}{\partial \tilde{x}_{i,j}}.$$

Furthermore, since the left hand side of (26) belongs to $\mathfrak{a}_j R[\tilde{x}]$, all $\partial(\nabla_u f)/\partial \tilde{x}_{i,j}$ must be zero for $u < i$, by the choice of basis (see Remark 7.11). Applied with $f = f_v = x_v + g_v$, the claim follows from this. \square

7.13. *Remark.* The section of $\nabla_Z X \rightarrow X$ given in the proof of the theorem is induced by ∇_0 , and will be called the *canonical section* of $\nabla_Z X \rightarrow X$. We will identify X as a closed subscheme of $\nabla_Z X$ via this canonical section.

7.14. **Corollary.** *If X is a d -dimensional affine variety and Z an Artinian local scheme of length l , then $\nabla_Z X$ is an ld -dimensional variety.*

Proof. Since X is irreducible, it contains a dense open subset U which is non-singular. By Theorem 7.12, the pull-back $\nabla U = U \times_X \nabla X$ is a dense open subset of ∇X . Moreover, Theorem 7.12 also yields that $\nabla U \rightarrow U$ is smooth. Since U is non-singular, so is therefore ∇U . In particular, ∇U is irreducible, whence so is ∇X . Moreover, by the proof of the theorem, U is defined by an ideal of height $n - d$, and ∇U by an ideal of height $l(n - d)$. Hence ∇U has dimension $ln - l(n - d) = ld$, whence so does ∇X . \square

7.15. **Corollary.** *If $\bar{Z} \subseteq Z$ is a closed immersion of Artinian local F -schemes, then for any affine F -scheme X , we have a split surjection $\nabla_Z X \rightarrow \nabla_{\bar{Z}} X$, making the diagram*

$$(27) \quad \begin{array}{ccc} & \nabla_Z X & \\ & \swarrow \quad \searrow & \\ \nabla_Z X & \xrightarrow{\quad} & X \end{array}$$

commute.

Proof. Let ϕ be the schemic formula defining X , and let $A := F[x]/I(\phi)$ be its coordinate ring, with x an n -tuple of variables. Let R and \bar{R} be the corresponding Artinian local rings, and let l and \bar{l} be their respective lengths. Let \mathfrak{a} be the kernel of the epimorphism $R \rightarrow \bar{R}$. We may choose a basis $\Delta = \{\alpha_0, \dots, \alpha_{l-1}\}$ of R such that $\alpha_j \in \mathfrak{a}$ for $j \geq \bar{l}$. Hence the images of the first \bar{l} elements of Δ form a basis $\bar{\Delta}$ of \bar{R} . In view of Lemma 7.10, we may use these two bases to calculate $\nabla_R \phi$ and $\nabla_{\bar{R}} \phi$. Let \tilde{x} be the ln -tuple of arc variables. Let us denote the $\bar{l}n$ first variables in \tilde{x} by \bar{x} , and let J be the ideal generated by the remaining variables. Hence $\tilde{A} := F[\tilde{x}]/I(\nabla_R \phi)$ and $\bar{A} := F[\bar{x}]/I(\nabla_{\bar{R}} \phi)$ are the respective coordinate rings of $\nabla_Z X$ and $\nabla_{\bar{Z}} X$. For $f \in F[x]$, equation (22) shows that $\nabla_i f$ belongs to $F[\bar{x}]$ for $i < \bar{l}$, and belongs to J for $i \geq \bar{l}$. Hence the embedding $F[\bar{x}] \rightarrow F[\tilde{x}]$ induces a homomorphism $\bar{A} \rightarrow \tilde{A}$. Moreover, the isomorphism $\tilde{A}/J\tilde{A} \cong \bar{A}$ shows that this homomorphism is split. \square

Note that if we let $\bar{Z} = \text{Spec } F$ be the closed subscheme given by the residue field, then $\nabla_{\bar{Z}} X = X$, showing that $\nabla_Z X \rightarrow X$ is induced by the residue map.

7.16. **Corollary.** *Each Artinian F -scheme Z induces an endomorphism ∇_Z , called the arc map along Z , on $\mathbf{Gr}(F^{\text{sch}})$ (respectively, on $\mathbf{Gr}(F^{\text{pp}})$), by sending a class $[\phi]$ to the class $[\nabla_Z \phi]$.*

Proof. Let us show in general that if θ is a morphic formula giving a morphism $\phi \rightarrow \psi$, then $\nabla \theta$ is also morphic and induces a morphism $\nabla \phi \rightarrow \nabla \psi$. We verify this on an arbitrary F -algebra A . Let $a \in \nabla \phi(A)$ and put $\tilde{a} := \pi^{-1}(a)$, so that $\tilde{a} \in \phi(\tilde{A})$, where $\tilde{A} := R \otimes_F A$ and R is the Artinian coordinate ring of Z . Since θ is morphic, it satisfies the morphic conditions (1)–(3), and hence we can find $\tilde{b} \in \psi(\tilde{A})$, such that $(\tilde{a}, \tilde{b}) \in \theta(\tilde{A})$. Let $b := \pi(\tilde{b})$ so that $b \in \nabla \psi(A)$ and $(a, b) \in \nabla \theta(A)$. In particular, a satisfies the morphic conditions (1)

and (3), and by a similar, easy argument we also verify (2), showing that $\nabla\theta$ defines a morphism $\nabla\phi \rightarrow \nabla\psi$.

Since ∇ preserves explicit and pp-formulae by Lemma 7.10 and its proof, our previous argument then shows that it also preserves *Sch*-isomorphisms. So remains to verify that ∇ also preserves scissor relations. This is clear, however, since $\nabla(\phi \vee \psi) = \nabla\phi \vee \nabla\psi$ and $\nabla(\phi \wedge \psi) = \nabla\phi \wedge \nabla\psi$ by the proof of Proposition 7.4. This also shows that ∇ preserves multiplication, showing that it is a ring endomorphism on either Grothendieck ring. \square

Integration. Integration is derived from the arc map as follows. Let X and Z be affine F -schemes, with Z Artinian. We define their *arc-integral* as the class

$$\int Z dX := [\nabla_Z X]$$

in $\mathbf{Gr}(F^{\text{sch}})$ (or in $\mathbf{Gr}(F^{\text{pp}})$). Corollary 7.16 shows that this is well-defined.

7.17. Proposition. *The arc-integral $\int Z dX$ only depends on the class of the Artinian scheme Z in $\mathbf{Gr}_0(F^{\text{sch}})$ and the class of the F -scheme X in $\mathbf{Gr}(F^{\text{sch}})$. In particular, for any $p \in \mathbf{Gr}(F^{\text{sch}})$ and $q \in \mathbf{Gr}_0(F^{\text{sch}})$, the arc-integral $\int q dp$ is a well-defined element of $\mathbf{Gr}(F^{\text{sch}})$.*

The same result holds upon replacing $\mathbf{Gr}(F^{\text{sch}})$ by $\mathbf{Gr}(F^{\text{pp}})$ everywhere.

Proof. The dependence on the class of X follows from Corollary 7.16. Using Corollary 4.9, one easily shows that $\int Z dX$ only depends on the isomorphism class of Z . To show that it even depends on the class of Z in $\mathbf{Gr}_0(F^{\text{sch}})$ only, we need to show that it vanishes on any scissor relation. By Lemma 3.8 (using Lemmas 4.2 and 4.4), it suffices to verify this on the scissor relation $Z + Z' - Z \sqcup Z'$, with Z and Z' Artinian schemes. We need to show that

$$(28) \quad \int Z dX + \int Z' dX = \int Z \sqcup Z' dX.$$

Let R and R' be the corresponding Artinian F -algebras, and let ϕ be the schemic formula defining X . Hence $R \oplus R'$ is the Artinian algebra corresponding to $Z \sqcup Z'$. Therefore, (28) is equivalent with showing that $[\nabla_R \phi] + [\nabla_{R'} \phi] = [\nabla_{R \oplus R'} \phi]$, and this, in turn will follow if we can show that the disjoint sum $\nabla_R \phi \oplus \nabla_{R'} \phi$ is isomorphic with $\nabla_{R \oplus R'} \phi$ modulo \mathbf{Art}_F . We verify this on an arbitrary model S of \mathbf{Art}_F . Using Lemma 3.3 and Proposition 7.4 both twice, we have

$$\begin{aligned} \nabla_{R \oplus R'} \phi(S) &= \phi((R \oplus R') \otimes_F S) \\ &= \phi((R \otimes_F S) \oplus (R' \otimes_F S)) \\ &\cong \phi(R \otimes_F S) \sqcup \phi(R' \otimes_F S) \\ &= (\nabla_R \phi)(S) \sqcup (\nabla_{R'} \phi)(S) \\ &= (\nabla_R \phi \oplus \nabla_{R'} \phi)(S), \end{aligned}$$

where the middle bijection is induced by the canonical bijection between $(R \oplus R')^n$ and $R^n \oplus (R')^n$. Since this bijection is easily seen to be induced by an (explicit) isomorphism, we completed the proof of (28).

The same argument shows that $\int \cdot dX$ is additive, and hence can be defined on any element $q \in \mathbf{Gr}_0(F^{\text{sch}})$. By Lemma 5.3, we can write an arbitrary element $p \in \mathbf{Gr}(F^{\text{sch}})$ as a difference $[X] - [X']$. We let

$$\int q dp := \int q dX - \int q dX'.$$

This is well-defined, as each arc map ∇_Z is a ring homomorphism, whence in particular additive. \square

Since $[\mathrm{Spec} F] = 1$ and the arc map $\nabla_{\mathrm{Spec} F}$ is the identity, we get the suggestive formula

$$\int dX = [X].$$

Since arc maps are also multiplicative, we get the following Fubini-type formula:

7.18. Proposition. *For Z an Artinian F -scheme and X and Y affine F -schemes, we have*

$$\int Z d(X \times_F Y) = \int Z dX \cdot \int Z dY$$

in $\mathbf{Gr}(F^{\mathrm{sch}})$. \square

The same product formula also holds in $\mathbf{Gr}(F^{\mathrm{pp}})$, where we may even drop the affineness assumption in view of Theorem 6.4. Since the arc formula of the Lefschetz formula in one variable is the Lefschetz formula in $\ell(R)$ variables, we immediately obtain from Lemma 7.3 that:

7.19. Proposition. *For any element $q \in \mathbf{Gr}_0(F^{\mathrm{sch}})$, we have $\int q d\mathbb{L} = \mathbb{L}^{\ell(q)}$.* \square

The next result generalizes this to an arbitrary smooth scheme, provided we work in the Grothendieck ring $\mathbf{Gr}(F^{\mathrm{pp}})$; this is needed since we need the covering properties proven in Theorem 6.4. We call a morphism $Y \rightarrow X$ of F -schemes a *locally trivial fibration with fiber W* if for each (closed) point $P \in X$, we can find an open $U \subseteq X$ containing P such that the restriction of $Y \rightarrow X$ to U is isomorphic with the projection $U \times_F W \rightarrow U$.

7.20. Corollary. *If $\bar{Z} \subseteq Z$ is a closed immersion of Artinian local F -schemes, and X is a smooth d -dimensional affine F -scheme, then $\nabla_Z X \rightarrow \nabla_{\bar{Z}} X$ is a locally trivial fibration with fiber \mathbb{A}_F^{dm} , where $m = \ell(Z) - \ell(\bar{Z})$. In particular,*

$$\int Z dX = [X] \cdot \mathbb{L}^{d\ell(Z)-d}$$

in $\mathbf{Gr}(F^{\mathrm{pp}})$.

Proof. Let R and \bar{R} be the Artinian local coordinate rings of Z and \bar{Z} respectively, and let ϕ be the schematic formula $f_1 = \cdots = f_s = 0$ defining $X \subseteq \mathbb{A}_F^m$. We will write ∇ and $\bar{\nabla}$ for the respective arc maps ∇_Z and $\nabla_{\bar{Z}}$, and similarly, π and $\bar{\pi}$ for the isomorphisms $R \cong F^{\ell(R)m}$ and $\bar{R} \cong F^{\ell(\bar{R})m}$. Since the composition of locally trivial fibrations is again a locally trivial fibration, with general fiber the product of the fibers, we may reduce to the case that $\bar{R} = R/\alpha R$ with α an element in the socle of R , that is to say, such that $\alpha m = 0$, where m is the maximal ideal of R . Let l be the length of R , and let Δ be a basis of R as in Remark 7.11, with $\alpha_{l-1} = \alpha$ (since α is a socle element, such a basis always exists). In particular, $\Delta - \{\alpha\}$ is a basis of \bar{R} . We will use these bases to calculate both arc maps.

We start with calculating a general fiber of the map $\nabla X \rightarrow \bar{\nabla} X$. By Corollary 4.9, it suffices to do this in an arbitrary model of \mathbf{Art}_F , that is to say, to calculate the fiber of an S -rational point on ∇X , where S is any Artinian local F -algebra. Let \bar{a} be an m -tuple in $\phi(\bar{R} \otimes_F S)$. By Proposition 7.5, its image $\bar{\pi}(\bar{a})$ is an $(l-1)m$ -tuple in $\bar{\nabla}\phi(S)$, corresponding by Lemma 4.5, therefore, to an S -rational point \bar{Q} of $\bar{\nabla} X$, and any S -rational point of ∇X is obtained in this way. Let P be the S -rational point on X given

as the image of \bar{Q} under the canonical morphism $\bar{\nabla}X \rightarrow X$.⁵ Hence $a := \pi_0(\bar{a})$ is the m -tuple in $\phi(S)$ corresponding to P .

The surjection $R \rightarrow \bar{R}$ induces a surjection $R \otimes_F S \rightarrow \bar{R} \otimes_F S$ and hence a map $\phi(R \otimes_F S) \rightarrow \phi(\bar{R} \otimes_F S)$. The fiber above \bar{a} is therefore defined by the equations $f_j(\bar{a} + \tilde{x}_{l-1}\alpha) = 0$, for $j = 1, \dots, s$. By Taylor expansion, this becomes

$$(29) \quad 0 = f_j(\bar{a} + \tilde{x}_{l-1}\alpha) = \left(\sum_{i=1}^m \frac{\partial f_j}{\partial x_i}(\bar{a}) \tilde{x}_{l-1,i} \right) \alpha$$

since $f_j(\bar{a}) = 0$ and $\alpha^2 = 0$ in $R \otimes_F S$. In fact, since $\bar{a} \equiv a \pmod{\mathfrak{m}(R \otimes_F S)}$ and $\alpha \mathfrak{m} = 0$, we may replace each $\partial f_j / \partial x_i(\bar{a})$ in (29) by $\partial f_j / \partial x_i(a)$. In terms of S -rational points, therefore, the fiber above \bar{Q} is the linear subspace of S^m defined as the kernel of the Jacobian $(s \times n)$ -matrix $\text{Jac}_\phi(P)$.

Since X is non-singular at P , this matrix has rank $m - d$, and hence its kernel is a d -dimensional linear subspace. This proves that the fiber is equal to \mathbb{A}_S^d . More precisely, we may choose ϕ so that the first $(m - d) \times (m - d)$ -minor in $\text{Jac}_\phi(P)$ is invertible. Therefore, by Kramer's rule, we may express each $\tilde{x}_{l-1,i}$ with $i \leq m - d$ as a linear combination of the $\tilde{x}_{l-1,j}$ with $j > m - d$ on an open neighborhood of \bar{Q} . Since this holds in any S -rational point \bar{Q} of $\bar{\nabla}X$, we showed that h is a locally trivial fibration with fiber \mathbb{A}_F^d .

Applying this to $\nabla X \rightarrow X$, (note that $X = \nabla_F X$) we get a locally trivial fibration with fiber equal to $\mathbb{A}_F^{d(l-1)}$. The last assertion then follows from Lemma 7.21 below. \square

7.21. Lemma. *If $f: Y \rightarrow X$ is a locally trivial fibration of F -schemes with fiber Z , then $[Y] = [X] \cdot [Z]$ in $\mathbf{Gr}(F^{\text{pp}})$.*

Proof. By definition and compactness, there exists a finite open covering $\{U_1, \dots, U_m\}$ of X , so that

$$f^{-1}(U_i) \cong U_i \times_F Z,$$

for $i = 1, \dots, m$. Taking classes in $\mathbf{Gr}(F^{\text{pp}})$, Lemma 5.5 yields $[f^{-1}(U_i)] = [U_i] \cdot [Z]$. Since the $f^{-1}(U_i)$ form an open affine covering of Y , Theorem 6.4 yields, after taking the m -th scissor polynomial on both sides,

$$[Y] = [S(f^{-1}(U_1), \dots, f^{-1}(U_m))] = [S(U_1, \dots, U_m)] \cdot [Z] = [X] \cdot [Z]$$

in $\mathbf{Gr}(F^{\text{pp}})$. \square

7.22. Remark. Example 7.7 shows that over a singular point, the dimension of the fiber may increase.

Igusa-zeta series. By a *germ*, we mean a pair (X, P) with X an F -scheme and P a closed point on X ; if X is a closed subscheme of X^* , then we also say that (X, P) is a germ in X^* . For any F -scheme Y , we can define the (*geometric*) *Igusa-zeta series of Y along the germ (X, P)* as the formal power series

$$\text{Igu}_Y^{(X,P)}(t) := \int \text{Hilb}_P(X) dY = \sum_n \left(\int J_P^n X dY \right) t^n = \sum_n [\nabla_{J_P^n X} Y] \cdot t^n$$

in $\mathbf{Gr}(F^{\text{sch}})[[t]]$. Note that this is well-defined since each jet is Artinian. This definition generalizes the one in [5] or [7, §4]:

⁵The image of an S -rational point $\text{Spec } S \rightarrow Y$ under a morphism $Y \rightarrow X$ is simply the composition of these two morphisms, yielding an S -rational point on X .

7.23. Proposition. *The Igusa-zeta series $\text{Igu}_Y^{(\mathbb{A}_F^1, O)}$ of Y along the germ of the origin on the affine line is sent under the canonical homomorphism*

$$\mathbf{Gr}(F^{\text{sch}})[[t]] \rightarrow \mathbf{Gr}(F^{\text{var}})[[t]]$$

to the geometric Igusa-zeta function $\text{Igu}_Y^{\text{geom}}$ of Y . If F has characteristic zero, then this image is a rational function.

Proof. Let P be the origin on the affine line. By our discussion preceding Example 7.7, the arc-integral $\int J_P^n \mathbb{L} dY$ is equal to the class of the n -th truncated arc space $\mathcal{L}_n(Y)$, and hence the first assertion follows from the definition of the geometric Igusa-zeta function in [7, §4]. Rationality over the classical Grothendieck ring $\mathbf{Gr}(F^{\text{var}})$ is proven in [7, Theorem 4.2.1]. \square

For curves, we can give an explicit formula for the Igusa-zeta series of the Lefschetz class:

7.24. Proposition. *If (C, P) is a germ of a point of multiplicity e on a curve C over F , then*

$$\text{Igu}_{\mathbb{A}_F^1}^{(C, P)} = \frac{p(t)}{1 - \mathbb{L}^e t}$$

for some polynomial $p \in \mathbf{Gr}(F^{\text{sch}})[t]$.

Proof. By definition, the multiplicity of the germ (C, P) is the multiplicity of the local ring $\mathcal{O}_{C, P}$. By Hilbert theory, there exist $b, N \in \mathbb{Z}$ such that the length of $J_P^n C$ for $n \geq N$ is equal to $en + b$. Using Proposition 7.19, we get

$$\int J_P^n C d\mathbb{L} = \mathbb{L}^{en+b}$$

for $n \geq N$. Hence $\text{Igu}_{\mathbb{A}_F^1}^{(C, P)}$ is the sum of some polynomial of degree N and the power series

$$\sum_n \mathbb{L}^{en+b} t^n = \frac{\mathbb{L}^b}{1 - \mathbb{L}^e t},$$

from which the assertion easily follows. \square

The above proof shows that

$$(30) \quad \text{Igu}_{\mathbb{A}_F^1}^{(X, P)} = \sum_n \mathbb{L}^{j_P^n(X)} t^n$$

for any germ (X, P) , where $j_P^n(X) := \ell(J_P^n X)$. For a smooth scheme, we have the following rationality result:

7.25. Proposition. *Let (C, P) be a germ of multiplicity e on a curve. For any d -dimensional smooth affine scheme Y over F , the Igusa-zeta series of Y along the germ (C, P) is a rational function over $\mathbf{Gr}(F^{\text{pp}})$. More precisely,*

$$(31) \quad \text{Igu}_Y^{(C, P)} = \frac{p(t)}{1 - \mathbb{L}^{de} t}$$

for some polynomial $p \in \mathbf{Gr}(F^{\text{pp}})[t]$.

Proof. By Hilbert theory, there exist $b, N \in \mathbb{Z}$ such that $j_P^n(C) = en + b$ for $n \geq N$. By Corollary 7.20, the coefficient of the n -th term in $\text{Igu}_Y^{(C, P)}$ is therefore equal to $[Y] \cdot \mathbb{L}^{d(en+b-1)}$ for $n \geq N$, and the result follows as in the proof of Proposition 7.24. \square

Motivic integrals. If we integrate over a higher dimensional scheme, then (30) suggests that we should calibrate the Igusa-zeta series to maintain rationality. To this end, we will define a normalized integration, which more closely resembles the motivic integration of Kontsevich, Denef and Loeser ([4, 6, 12]). More precisely, let $\mathbf{Gr}(F^{\text{sch}})_{\mathbb{L}}$ and $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}}$ be the respective localization of $\mathbf{Gr}(F^{\text{sch}})$ and $\mathbf{Gr}(F^{\text{pp}})$ at \mathbb{L} , that is to say, $\mathbf{Gr}(F^{\text{sch}})_{\mathbb{L}} = \mathbf{Gr}(F^{\text{sch}})[\mathbb{L}^{-1}]$ and $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}} = \mathbf{Gr}(F^{\text{pp}})[\mathbb{L}^{-1}]$. Let X and Z be affine F -schemes with Z Artinian. We define the *motivic integral of X along Z* to be

$$\int^{\text{mot}} Z dX := \mathbb{L}^{-dl} \cdot \int Z dX = \mathbb{L}^{-dl} \cdot [\nabla_Z X]$$

with d the dimension of X and l the length of Z , viewed either as an element in $\mathbf{Gr}(F^{\text{sch}})_{\mathbb{L}}$ or $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}}$.

We define the *motivic Igusa-zeta series of Y along a germ (X, P)* as the formal power series

$$\text{Igu}_{Y_{\text{mot}}}^{(X,P)}(t) := \int^{\text{mot}} \text{Hilb}_P(X) dY = \sum_n \mathbb{L}^{-d \cdot j_P^n(X)} \left(\int J_P^n X dY \right) t^n$$

in $\mathbf{Gr}(F^{\text{sch}})_{\mathbb{L}}[[t]]$ (respectively, in $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}}[[t]]$), where d is the dimension of Y . In particular, by the same argument as in the proof of Proposition 7.25, we get

$$\text{Igu}_{Y_{\text{mot}}}^{(X,P)} = \frac{[Y] \cdot \mathbb{L}^{-d}}{1-t},$$

over $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}}$, for any germ (X, P) , and any smooth affine F -scheme Y . This raises the following question:

7.26. Conjecture. *If F is an algebraically closed field of characteristic zero, then, for any affine F -scheme Y , the motivic Igusa-zeta series $\text{Igu}_{Y_{\text{mot}}}^{(X,P)}$ of Y along an arbitrary germ (X, P) is rational over $\mathbf{Gr}(F^{\text{pp}})_{\mathbb{L}}$.*

8. INFINITARY GROTHENDIECK RINGS

In this section, we will extend the previous definitions to include infinitary formulae. This turns out to be necessary when dealing with the complement of an open subscheme, as we mentioned already in the introduction.

Formularies. Let \mathcal{L} be an arbitrary first-order language, and let Φ be a collection of \mathcal{L} -formulae of some fixed arity n , which we then call the *arity* of Φ . For an \mathcal{L} -structure M , let $\Phi(M)$ be the subset in M^n given as the union of all $\phi(M)$ with $\phi \in \Phi$. In other words, we view Φ as an infinitary disjunction, defining in each structure a subset $\Phi(M)$ which is in general only infinitary definable (and its complement is type-definable). We call $\Phi(M)$ the *interpretation of Φ in M* .

Since elementary classes do no longer behave the same as their theories on infinitary formulae, we must shift our attention from the latter to the former. So, let \mathfrak{K} be a class of \mathcal{L} -structures, and let Φ and Ψ be two collections of \mathcal{L} -formulae of the same arity n . We say that Φ and Ψ are *\mathfrak{K} -equivalent*, if $\Phi(M) = \Psi(M)$ for all $M \in \mathfrak{K}$. In particular, if we let Φ^\vee be the collection of all finite disjuncts of formulae in Φ , then Φ^\vee is \mathfrak{K} -equivalent with Φ , and so without loss of generality, we may always assume, up to equivalence, that a collection is closed under finite disjunctions.

We say that Φ is a *formulary with respect to \mathfrak{K}* , if for each structure $M \in \mathfrak{K}$, there is some $\phi \in \Phi$ such that $\phi(M) = \Phi(M)$. Note that ϕ will in general depend on the structure M . Put differently, although in each \mathfrak{K} -structure M , the subset $\Phi(M)$ is definable, its definition

depends on the given structure M . We say that Φ is *first-order*, if it is \mathfrak{K} -equivalent with a first-order formulae ϕ . Although we do not insist in this definition on ϕ being part of Φ , there is no loss of generality including it, since adding it yields an \mathfrak{K} -equivalent formulary. In particular, we may view formulae as first-order formularies and we will henceforth identify both (up to \mathfrak{K} -equivalence).

Most of the logical operations generalize to this infinitary setting. Namely, given $\Phi \subseteq \mathcal{L}_n$ and $\Psi \subseteq \mathcal{L}_m$, let $\Phi \wedge \Psi$ be the collection of all $\phi \wedge \psi$ with $\phi \in \Phi$ and $\psi \in \Psi$, and let $\Phi \vee \Psi := \Phi \cup \Psi$. Similarly, we define $\Phi \times \Psi \subseteq \mathcal{L}_{n+m}$ as the collection of all $\phi \times \psi$. In case \mathcal{L} contains the two constant symbols 0 and 1, we also define $\Phi \oplus \Psi$ as $\Phi' \cup \Psi'$ where Φ' consists of all $\phi \wedge (v_{n+1} = 0)$ for $\phi \in \Phi$, and Ψ' of all $\psi \wedge (v_{n+1} = 1)$ for $\psi \in \Psi$ (where we assume $m \leq n$). We leave it to the reader to verify that all these operations preserve formularies. If a formulary is not first-order, then we can, however, not define its negation (in model-theoretic terms, the negation of a formulary is a type).

From now on, $\mathcal{L} := \mathcal{L}_F$, the language of F -algebras, for F an algebraically closed field, and $\mathfrak{K} = \mathbf{Art}_F$ the collection of all Artinian local F -algebras. We call a formulary Φ respectively *schemic* or *pp* if all formulae in Φ are of that kind.

Total jets and formal schemes. Let ϕ be an arbitrary formula and ζ a schemic formula. We define the *total jet of ϕ along ζ* as the collection $J_\zeta \phi$ of all n -jets $J_\zeta^n \phi$. Let us show that $J_\zeta \phi$ is a formulary. Let $\mathfrak{a} := I(\zeta)$ be the ideal of ζ , and let (R, \mathfrak{m}) be an Artinian local F -algebra of length l . I claim that $(J_\zeta \phi)(R)$ is equal to $(J_\zeta^l \phi)(R)$. To prove this, it suffices to show that $(J_\zeta^n \phi)(R) = (J_\zeta^{n+1} \phi)(R)$, for all $n \geq l$. One direction is clear, so assume that $a \in (J_\zeta^{n+1} \phi)(R)$. Hence, for each $f \in \mathfrak{a}$, we have $f^{n+1}(a) = 0$, whence $f(a) \in \mathfrak{m}$. Since this holds for all $f \in \mathfrak{a}$, we see that $g(a) = 0$ in R for every $g \in \mathfrak{a}^n$, as $\mathfrak{m}^n = 0$. In conclusion, $a \in (J_\zeta^n \phi)(R)$.

If ϕ and ζ are schemic with $\zeta \Rightarrow \phi$, therefore corresponding to a closed immersion $Z \subseteq X$, then we also will write $J_Z X$ for $J_\zeta \phi$. We can give the following geometric interpretation of total jets:

8.1. Proposition. *For $Y \subseteq X$ a closed immersion of affine F -schemes, there is, for every Artinian local F -algebra R , a one-one correspondence between the R -rational points of the formal scheme \widehat{X}_Y and the interpretation of the formulary $J_Y X$ in R .*

Proof. Let $A := \mathcal{O}_X(X)$ be the coordinate ring of X , and I the ideal defining Y , that is to say, $\mathcal{O}_Y(Y) = A/I$. Hence $J_Y^n X = \text{Spec}(A/I^n)$. By [9, II.§9], the formal completion \widehat{X}_Y is the ringed space with underlying set equal to the underlying set of Y and with sheaf of rings the inverse limit of the sheafs $\mathcal{O}_{J_Y^n X}$. In particular, the ring of global sections is equal to the I -adic completion \widehat{A} of A .

Let (R, \mathfrak{m}) be an Artinian local F -algebra, and let $\text{Spec } R \rightarrow \widehat{X}_Y$ be an R -rational point of \widehat{X}_Y over F . Taking global sections, we get a F -algebra homomorphism $\widehat{A} \rightarrow R$. Since the closed point of $\text{Spec}(R)$ is sent to a point in the underlying set of Y , we have $IR \subseteq \mathfrak{m}$. If R has length l , then I^l lies in the kernel of $\widehat{A} \rightarrow R$, and so we get a factorization $\widehat{A} \rightarrow \widehat{A}/I^l \cong A/I^l \rightarrow R$, that is to say, an R -rational point $\text{Spec } R \rightarrow J_Y^l X$. By Lemma 4.5, this corresponds to a tuple in $(J_Y^l X)(R)$ whence in $(J_Y X)(R)$, where, as before, we identify jets with the schemic formulae defining them. Conversely, a tuple in $(J_Y X)(R)$ lies in some $(J_Y^n X)(R)$, and hence, by Lemma 4.5 yields a F -algebra homomorphism $A/I^n \rightarrow R$. Composition with the canonical surjection $\widehat{A} \rightarrow A/I^n$ then induces an R -rational point on \widehat{X}_Y . \square

This proposition allows us to identify the total jet $J_Y X$ with the formal scheme \widehat{X}_Y , which we henceforth will do.

8.2. Lemma. *If $Y_i \subseteq X_i$ for $i = 1, 2$ are two closed immersions, then we have the following product formula*

$$(J_{Y_1} X_1) \times_F (J_{Y_2} X_2) \cong J_{Y_1 \times_F Y_2} (X_1 \times_F X_2).$$

Proof. Let A_i be the coordinate ring of X_i , and I_i the ideal defining Y_i . Hence $X_1 \times X_2$ has coordinate ring $A := A_1 \otimes_F A_2$ and $Y_1 \times Y_2$ is defined by the ideal $I := I_1 A + I_2 A$. The ideals corresponding to the schemic formulae in the total jets $J_{Y_i} X_i$ and $J_{Y_1 \times Y_2} (X_1 \times X_2)$ are respectively the powers of I_i and of I . Since $I^{2n} \subseteq I_1^n A + I_2^n A \subseteq I^n$, the formula follows readily. \square

Although one could give a more general notion of morphism based on morphic formulae, we define them only using (first-order) formulae. Namely, let \mathcal{I} be a family of (first-order) formulae, and let $\Phi \subseteq \mathcal{L}_n$ and $\Psi \subseteq \mathcal{L}_m$ be formulae. By an \mathcal{I} -morphism $f: \Phi \rightarrow \Psi$, we mean a morphic formula $\theta \in \mathcal{I}$ such that for each Artinian local F -algebra R , the definable subset $\theta(R) \subseteq R^{n+m}$ restricted to $\Phi(R)$ is the graph of a map $f_R: \Phi(R) \rightarrow \Psi(R)$. Here we have to replace the morphic conditions (1)–(3) by their appropriate counterparts (more precisely, replace ϕ and ψ in these sentences respectively by the infinite disjunctions $\bigvee \Phi$ and $\bigvee \Psi$; then require that the resulting (non-first order) sentences to hold in any Artinian local F -algebra). As before, an \mathcal{I} -isomorphism is an \mathcal{I} -morphism which is a bijection on each model, and whose inverse is also an \mathcal{I} -morphism.

The infinitary pp-Grothendieck ring. Let \mathcal{PP}^∞ be the lattice of all formulae consisting of formulae in \mathcal{PP} , with \wedge and \vee as defined above. On this lattice, we have an isomorphism relation \cong_{Sch} , given by schemic isomorphisms modulo \mathbf{Art}_F . We define the *infinitary pp-Grothendieck ring* as

$$\mathbf{Gr}^\infty(F^{\text{pp}}) := \mathbf{K}_{\cong_{Sch}}(\mathcal{PP}^\infty),$$

where the multiplication is induced by the same argument as in Lemma 3.4 by the multiplication on formulae. Recall that $\mathbf{Gr}^\infty(F^{\text{pp}})$ is the quotient of the free Abelian group $\mathbb{Z}[\mathcal{PP}^\infty]$ modulo the subgroup generated by all $\langle \Phi \rangle - \langle \Psi \rangle$ for Sch -isomorphic formulae Φ and Ψ , and by all $\langle \Phi \vee \Psi \rangle + \langle \Phi \wedge \Psi \rangle - \langle \Phi \rangle - \langle \Psi \rangle$. Note that \mathcal{PP}^∞ is not Boolean, since the negation of a formula does not exist. In particular, we do no longer have the analogue of the second property in Lemma 3.7. I do not know whether the analogue of Corollary 3.9 holds (the proof of the corollary relies on the negation property, whence is not admissible here).

Since pp-formulae are just first-order pp-formulae, we get a canonical homomorphism

$$\mathbf{Gr}(F^{\text{pp}}) \rightarrow \mathbf{Gr}^\infty(F^{\text{pp}}).$$

This homomorphism, however, is not an embedding, as can be seen from the following relation.

8.3. Theorem. *For $Y \subseteq X$ a closed immersion of affine F -schemes, we have a relation*

$$[X] = [X - Y] + [J_Y X].$$

in $\mathbf{Gr}^\infty(F^{\text{pp}})$.

Proof. Let ϕ be the schemic formula of X and A its coordinate ring. Let Y be defined by $\phi \wedge (f_1 = \dots = f_s = 0)$, whence $I := (f_1, \dots, f_s)A$ its ideal of definition in X . Let ψ_i be the pp-formula defining the basic open $D_i := \text{Spec } A_{f_i}$ of $U := X - Y$, that is to say, $\psi_i := \phi \wedge (\exists y)(yf_i = 1)$. It follows that $\{D_1, \dots, D_n\}$ is an open covering of U , and hence by Corollary 6.6, we have

$$[U] = [\psi_1 \vee \dots \vee \psi_n]$$

in $\mathbf{Gr}(F^{\text{pp}})$, whence also in $\mathbf{Gr}^\infty(F^{\text{pp}})$. So remains to show that $\phi \wedge \neg\psi_1 \wedge \dots \wedge \neg\psi_n$ is \mathbf{Art}_F -equivalent to the formulary $J_Y X$. One direction is obvious, and to verify the other, we check this in an arbitrary Artinian local F -algebra (R, \mathfrak{m}) . Let a be an n -tuple in R satisfying $\phi \wedge \neg\psi_1 \wedge \dots \wedge \neg\psi_n$. In particular, $f_i(a) \in \mathfrak{m}$, for all i . For l at least the length of R , we therefore get $g(a) = 0$ for every $g \in I^l$, from which it follows that $a \in (J_Y X)(R)$. \square

Using Proposition 8.1, we get the following more suggestive version of Theorem 8.3: for any closed immersion $Y \subseteq X$ of affine F -schemes, we have

$$(32) \quad [X] = [X - Y] + [\widehat{X}_Y]$$

in $\mathbf{Gr}^\infty(F^{\text{pp}})$.

Formal Lefschetz class. We define the *formal Lefschetz class*, denoted $\widehat{\mathbb{L}}$, as the class of the formal completion of the affine line at the origin O , that is to say,

$$\widehat{\mathbb{L}} := [J_O \mathbb{A}_F^1] = [\widehat{(\mathbb{A}_F^1)_O}].$$

By (32), we may now give the correct decomposition formula for the Lefschetz class discussed at the end of Remark 6.7, namely, in $\mathbf{Gr}^\infty(F^{\text{pp}})$ we have

$$(33) \quad \mathbb{L} = \mathbb{L}^* + \widehat{\mathbb{L}}.$$

By Lemma 8.2 and the Nullstellensatz, we have

$$(34) \quad [J_P \mathbb{A}_F^n] = [\widehat{(\mathbb{A}_F^n)_P}] = \widehat{\mathbb{L}}^n,$$

for any closed point P in \mathbb{A}_F^n . We next calculate the class of projective space. Using the standard affine covering by the basic opens $D(x_i) \subseteq \mathbb{P}_F^n$, one easily verifies that the morphism $(\mathbb{A}_F^{n+1} - O) \rightarrow \mathbb{P}_F^n$, given by sending the affine coordinates (x_0, \dots, x_n) to the projective ones $(x_0 : \dots : x_n)$, is a locally trivial fibration with fiber $\mathbb{A}_F^1 - O$, where O is the origin. By Theorem 8.3 and (34), the class of $\mathbb{A}_F^i - O$ in $\mathbf{Gr}^\infty(F^{\text{pp}})$ is equal to $\mathbb{L}^i - \widehat{\mathbb{L}}^i$, for every i . By Lemma 7.21 applied to this locally trivial fibration $\mathbb{A}_F^{n+1} - O \rightarrow \mathbb{P}_F^n$, we get

$$\mathbb{L}^{n+1} - \widehat{\mathbb{L}}^{n+1} = [\mathbb{P}_F^n] \cdot (\mathbb{L} - \widehat{\mathbb{L}}).$$

We would like to divide both sides by $\mathbb{L} - \widehat{\mathbb{L}}$, but a priori, this is not a zero-divisor in $\mathbf{Gr}^\infty(F^{\text{pp}})$. The resulting formula does hold, as we now calculate by a different method:

8.4. Proposition. *For each n , the class of projective n -space in $\mathbf{Gr}^\infty(F^{\text{pp}})$ is given by the formula*

$$[\mathbb{P}_F^n] = \sum_{m=0}^n \mathbb{L}^m \cdot \widehat{\mathbb{L}}^{n-m}.$$

Proof. Let $(x_0 : \cdots : x_n)$ be the homogeneous coordinates of \mathbb{P}_F^n , and let $U_i := D_+(x_i)$ be the basic open given as the complement of the x_i -hyperplane. Hence each U_i is isomorphic with \mathbb{A}_F^n and their union is equal to \mathbb{P}_F^n . Therefore,

$$(35) \quad [\mathbb{P}_F^n] = [S(U_0, \dots, U_n)]$$

in $\mathbf{Gr}(F^{\text{pp}})$ whence in $\mathbf{Gr}^\infty(F^{\text{pp}})$. So we need to calculate the class of each intersection occurring in the right-hand side scissor relation. One easily verifies that, for $m \geq 0$, any intersection of m different opens U_i is isomorphic to the open $\mathbb{A}_F^{n-m} \times (\mathbb{A}_F^*)^m$, where \mathbb{A}_F^* is the affine line minus a point. Since $[\mathbb{A}_F^*] = \mathbb{L}^* = \mathbb{L} - \widehat{\mathbb{L}}$ by (33), the class of such an intersection is equal to the product $\mathbb{L}^{n-m}(\mathbb{L} - \widehat{\mathbb{L}})^m$. Since there are $\binom{n+1}{m}$ terms of degree m in the scissor polynomial S_{n+1} , the class of \mathbb{P}_F^n is equal to $g(\mathbb{L}, \widehat{\mathbb{L}})$ by (35) and the previous discussion, with

$$g(t, u) := \sum_{m=0}^n (-1)^m \binom{n+1}{m} t^{n-m} (t-u)^m.$$

By the binomial theorem, $t^{n+1} - (t-u)g(t, u) = (t - (t-u))^{n+1} = u^{n+1}$, and hence

$$g(t, u) = \frac{t^{n+1} - u^{n+1}}{t-u} = \sum_{m=0}^n t^m u^{n-m},$$

as we wanted to show. \square

Although a priori an infinitary object, the infinitary pp-Grothendieck ring still specializes to the classical Grothendieck ring:

8.5. Proposition. *There exists a canonical homomorphism $\mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{var}})$.*

Proof. We use the following observation: if \mathfrak{K} is a finite collection of F -algebras, then any formulary is first-order modulo \mathfrak{K} . Indeed, let Φ be a formulary. As observed above, we may assume that it is closed under finite disjunctions. Let $\mathfrak{K} = \{R_1, \dots, R_s\}$, and let $\phi_i \in \Phi$ be such that $\Phi(R) = \phi_i(R)$. Hence Φ is \mathfrak{K} -equivalent with the (first-order) disjunction $\phi_1 \vee \dots \vee \phi_s$. We can apply this observation to the singleton $\{F\}$. Since \mathbf{ACF}_F , the theory of F , admits elimination of quantifiers, each class of a formulary in $\mathbf{Gr}^\infty(F^{\text{pp}})$ is equal to a class in $\mathbf{Gr}(F^{\text{var}})$. \square

8.6. Remark. From the proof it follows that image of the formal Lefschetz class $\widehat{\mathbb{L}}$ under the homomorphism $\mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{var}})$ is equal to 1.

Arc formularies. Given a formulary $\Phi \subseteq \mathcal{L}_n$, and an Artinian F -scheme Z , let us define $\nabla_Z \Phi$ as the formulary of all $\nabla_Z \phi$ with $\phi \in \Phi$. As the next result shows, in the formulary case, we are justified to call $\nabla_Z \Phi$ the *arc formulary of Φ along Z* .

8.7. Proposition. *If Z is a local Artinian F -scheme of length l with coordinate ring (R, \mathfrak{m}) , and $\Phi \subseteq \mathcal{L}_n$ an arbitrary formulary, then $\nabla_Z \Phi$ is also a formulary, and for any Artinian local F -algebra S , there is a one-one correspondence between $\Phi(R \otimes_F S)$ and $\nabla_Z \Phi(S)$ induced by the canonical isomorphism $\pi: R \otimes_F S \rightarrow S^l$.*

Moreover, this induces an arc map on the infinitary pp-Grothendieck ring $\mathbf{Gr}^\infty(F^{\text{pp}})$, and as before, we will write $\int Z d\Phi$ for the class of $\nabla_Z \Phi$.

Proof. We will prove the first two assertions simultaneously. Let (S, \mathfrak{n}) be an Artinian local F -algebra, put $\tilde{S} := R \otimes_F S$, and let $\tilde{\mathfrak{m}} := \mathfrak{m}\tilde{S} + \mathfrak{n}\tilde{S}$. Since $\tilde{S}/\tilde{\mathfrak{m}} \cong R/\mathfrak{m} \otimes_F S/\mathfrak{n} \cong F \otimes_F F = F$, as F is algebraically closed, \tilde{S} is local with maximal ideal $\tilde{\mathfrak{m}}$. In particular, \tilde{S} is an

Artinian local F -algebra, and hence there is some $\phi_0 \in \Phi$ such that $\phi_0(\bar{S}) = \Phi(\bar{S})$. Let a be an ln -tuple in $\nabla\Phi(S)$. Let \bar{a} be the n -tuple over \bar{S} such that $\pi(\bar{a}) = a$. By construction, $a \in \nabla\phi(S)$ for some $\phi \in \Phi$, and hence $\bar{a} \in \phi(\bar{S}) \subseteq \Phi(\bar{S}) = \phi_0(\bar{S})$ by definition of arc formulae. This in turn implies that $a \in \nabla\phi_0(S)$, showing that $\nabla\Phi(S) = \phi_0(S)$. The second assertion is then immediate by Proposition 7.4. The last assertion follows by the exact same argument as for Corollary 7.16, and its proof is left to the reader. \square

8.8. Corollary. *For (X, P) a germ in \mathbb{A}_F^n , and Z an Artinian local F -scheme of length l , we have*

$$\nabla_Z(J_P X) = (J_P X \times \mathbb{A}_F^{(l-1)n}) \cap \nabla_Z X,$$

where we view $J_P X$ as a closed subscheme of $\nabla_Z(J_P X)$ via the canonical section defined in Remark 7.13.

Proof. Suppose X is a closed subscheme of \mathbb{A}_F^n . An easy calculation shows that

$$(36) \quad J_P X = J_P \mathbb{A}_F^n \cap X.$$

By the Nullstellensatz, we may assume, without loss of generality, that P is the origin, and hence the ideals in $F[x]$ corresponding to the spheric formulae in $J_P \mathbb{A}_F^n$ are simply all the powers of the maximal ideal $(x_1, \dots, x_n)F[x]$. Let (R, \mathfrak{m}) be the Artinian local coordinate ring of Z , and let (S, \mathfrak{n}) be an arbitrary Artinian local F -algebra. As already remarked previously, $\bar{S} := R \otimes_F S$ is an Artinian local F -algebra with maximal ideal $\bar{\mathfrak{m}} := \mathfrak{m}\bar{S} + \mathfrak{n}\bar{S}$. An n -tuple \bar{a} over \bar{S} belongs to $J_P \mathbb{A}_F^n(\bar{S})$ if and only if all its entries are nilpotent, that is to say, if and only if $\bar{a} \in \bar{\mathfrak{m}}\bar{S}^n$. Since $\bar{a} \equiv \pi_0(\bar{a}) \pmod{\bar{\mathfrak{m}}\bar{S}}$, the latter condition is equivalent with $\pi_0(\bar{a}) \in \mathfrak{n}S^n$, which in turn is equivalent with $\pi_0(\bar{a}) \in J_P \mathbb{A}_F^n(S)$, showing that

$$\nabla_Z(J_P \mathbb{A}_F^n) = J_P \mathbb{A}_F^n \times \mathbb{A}_F^{(l-1)n}$$

under the identification from Remark 7.13. Taken together with (36) and the fact that the arc map preserves intersections, we get the desired equality. \square

8.9. Remark. It follows from the proof and (32) that we in fact have an equality

$$\nabla_Z(\widehat{X}_P) = ((\widehat{\mathbb{A}_F^n})_P \times \mathbb{A}_F^{(l-1)n}) \cap \nabla_Z X,$$

if (X, P) is a germ in \mathbb{A}_F^n .

We have the following analogue of Proposition 7.19 for the formal Lefschetz class:

8.10. Corollary. *For any element $q \in \mathbf{Gr}_0(F^{\text{sch}})$, we have $\int q d\widehat{\mathbb{L}} = \widehat{\mathbb{L}} \cdot \mathbb{L}^{\ell(q)-1}$.*

Proof. By additivity, it suffices to show this for q equal to an Artinian local scheme Z of length l . By Corollary 8.8, we have

$$\nabla_Z(J_O \mathbb{A}_F^1) = (J_O \mathbb{A}_F^1 \times \mathbb{A}_F^{l-1}) \cap \nabla_Z \mathbb{A}_F^1,$$

where O is the origin. Since $\nabla_Z \mathbb{A}_F^1 = \mathbb{A}_F^l$, taking classes therefore yields the asserted formula. \square

If instead we work in the Grothendieck ring, we may generalize the previous result to higher dimensional fibers:

8.11. Corollary. *Let $Y \subseteq X$ be a closed immersion of F -schemes, Z an Artinian F -scheme, and $\rho: \nabla_Z X \rightarrow X$ the canonical split projection. Then we have an equality*

$$[\nabla_Z(J_Y X)] = [J_{\rho^{-1}(Y)}(\nabla_Z X)]$$

in $\mathbf{Gr}^\infty(F^{\text{pp}})$.

Proof. By Theorem 8.3, we have an equality

$$[X] = [X - Y] + [J_Y X].$$

in $\mathbf{Gr}^\infty(F^{\text{pp}})$. Since $\nabla_Z \cdot$ is an endomorphism, we get

$$(37) \quad [\nabla_Z X] = [\nabla_Z(X - Y)] + [\nabla_Z(J_Y X)].$$

Since $X - Y \subseteq X$ is an open immersion, we have

$$\nabla_Z(X - Y) = \rho^{-1}(X - Y) = \nabla_Z X - \rho^{-1}(Y)$$

by (25) in Theorem 7.12. On the other hand, by another application of Theorem 8.3, we get

$$[\nabla_Z X] = [\nabla_Z X - \rho^{-1}(Y)] + [J_{\rho^{-1}(Y)} \nabla_Z X],$$

from which the assertion now follows immediately in view of (37). \square

9. GEOMETRIC IGUSA-ZETA SERIES OVER LINEAR ARCS

For various schemes X , we will calculate $\text{Igu}_X^{(\mathbb{A}, O)}$, that is to say, we want to calculate $\sum_n [\nabla_Z X] t^n$, where, for the remainder of this section $Z_n := \text{Spec}(F[\xi]/\xi^n F[\xi])$. To simplify notation, we simply write $\nabla_n X$ for the n -th linear arc scheme $\nabla_{Z_n} X$. We let $\rho_n: \nabla_n X \rightarrow X$ be the canonical split projection with section $X \hookrightarrow \nabla_n X$, and we view closed subschemes of X as closed subschemes of $\nabla_n X$ via the latter embedding.

The formal Grothendieck ring. In $\mathbf{Gr}^\infty(F^{\text{pp}})$, we define the *formal ideal* \mathfrak{N} as the ideal generated by the relations $[J_Y X] - [Y]$, for all closed immersions $Y \subseteq X$ of affine schemes. It follows that if $Y, Y' \subseteq X$ are two closed subschemes of X with the same underlying set, then $[Y] \equiv [Y'] \pmod{\mathfrak{N}}$, since they have the same total jets. Recall from Proposition 8.5 that we have a canonical homomorphism $\mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{var}})$. We prove below that \mathfrak{N} belongs to its kernel, and so we introduce the *formal Grothendieck ring* $\mathbf{Gr}(F^{\text{form}})$ as the quotient $\mathbf{Gr}^\infty(F^{\text{pp}})/\mathfrak{N}$.

9.1. Proposition. *We have a sequence of natural homomorphisms of Grothendieck rings $\mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{form}}) \rightarrow \mathbf{Gr}(F^{\text{var}})$.*

Proof. By definition of formulary, there exists a schematic formula ϕ in $J_Y X$ such that its F -rational points are given by $\phi(F)$, for a given closed immersion $Y \subseteq X$. In particular, $\phi(F)$ is equal to $Y(F)$, showing that $[\phi] = [Y]$ in $\mathbf{Gr}(F^{\text{var}})$. By the argument in Proposition 8.5, the image of $[J_Y X]$ in $\mathbf{Gr}(F^{\text{var}})$ is equal to $[\phi]$. This shows that \mathfrak{N} lies in the kernel of $\mathbf{Gr}^\infty(F^{\text{pp}}) \rightarrow \mathbf{Gr}(F^{\text{var}})$. \square

In particular, to prove the rationality of the geometric Igusa-zeta series over $\mathbf{Gr}(F^{\text{var}})_{\mathbb{L}}$, it will suffice to show that its image as a power series over $\mathbf{Gr}(F^{\text{form}})$ is rational over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$. We will simplify our notation and write $\text{Igu}_X^{\text{geom}}(t)$ for $\text{Igu}_X^{(\mathbb{A}, O)}$, viewed as a power series over $\mathbf{Gr}(F^{\text{form}})$.

9.2. Lemma. *If $\{Y_i\}_i$ is a constructible partition of a scheme X , then $[X] = \sum_i [Y_i]$ in $\mathbf{Gr}(F^{\text{form}})$.*

Proof. Note that this partition is finite, since X is Noetherian. Moreover, at least one part must be open, say Y_1 , and let $X_1 := X - Y_1$ be its complement. By Theorem 8.3, we have an equality $[X] = [Y_1] + [J_{X_1} X]$ in $\mathbf{Gr}^\infty(F^{\text{pp}})$. By definition of formal ideal, $[J_{X_1} X] \equiv [X_1]$ modulo \mathfrak{N} . Putting these two together, we get an identity $[X] = [Y_1] + [X_1]$ in $\mathbf{Gr}(F^{\text{form}})$. Moreover, $\{Y_2, Y_3, \dots\}$ is a constructible partition of X_1 , and so we are done by Noetherian induction. \square

9.3. Theorem. *Let X be a d -dimensional scheme, Y a closed subscheme containing the singular locus of X , and Z an Artinian scheme of length l . If $\rho: \nabla_Z X \rightarrow X$ denotes the canonical projection, then we have an equality*

$$[\nabla_Z X] = [X - Y] \cdot \mathbb{L}^{d(l-1)} + [\rho^{-1}(Y)]$$

in $\mathbf{Gr}(F^{\text{form}})$.

Proof. This follows immediately from Corollaries 7.20 and 8.11, but here is the argument in more detail: let us put $W := \nabla_Z X$ and $V := \rho^{-1}(Y)$. By Lemma 9.2, we have an equality $[W] = [W - V] + [V]$ in $\mathbf{Gr}(F^{\text{form}})$. Moreover, by Theorem 7.12, we have an isomorphism $W - V \cong \nabla_Z(X - Y)$. Since $X - Y$ is smooth by the choice of Y , $[\nabla_Z(X - Y)] = [X - Y] \cdot \mathbb{L}^{d(l-1)}$ by Corollary 7.20, and the assertion follows. \square

In order to simplify our notation, we will henceforth write X_s for the basic subset $D(s)$ in a scheme X , where s is a global section on X . Likewise, we write $X(s_1, \dots, s_n)$ for the intersection $X \cap V(s_1, \dots, s_n)$, for given global sections s_i . In this notation, we have, by Lemma 9.2, the following useful equality

$$(38) \quad [X] = [S_n(X_{s_1}, \dots, X_{s_n})] + [X(s_1, \dots, s_n)]$$

in $\mathbf{Gr}(F^{\text{form}})$. Note that products in the scissor polynomial are actually given by intersection, as in §2. For instance, (38) becomes for $n = 2$, the identity

$$[X] = [X_s] + [X_t] - [X_{st}] + [X(s, t)].$$

To derive the next identity, we introduce some further notation. Fix a scheme X and an n -tuple of global sections (s_1, \dots, s_n) . Given a *binary vector* δ of length n , that is to say, a n -tuple in $\{0, 1\}^n$, we will write X_δ for X_{s^δ} where s^δ is the product of all $s_i^{\delta_i}$, and we will write \bar{X}_δ for $X_\delta(s_\delta)$, where s_δ is the tuple of all s_i for which $\delta_i = 0$. In other words, \bar{X}_δ is defined by the formula expressing that each s_i is either a unit or zero, depending on whether δ_i is one or zero. One easily verifies that $X = \bigsqcup_\delta \bar{X}_\delta$, where δ runs over all binary vectors of length n . Applying Lemma 9.2 to this constructible partition, we get

$$(39) \quad [X] = \sum_\delta [\bar{X}_\delta]$$

in $\mathbf{Gr}(F^{\text{form}})$, where the sum runs over all binary vectors. Let us again illustrate this for $n = 2$, yielding the identity

$$[X] = [X_{st}] + [X_s(t)] + [X_t(s)] + [X(s, t)].$$

It is important to note that this equation is false in $\mathbf{Gr}^\infty(F^{\text{pp}})$, whence, in particular, we may not apply ∇_Z to it.

Before we turn to a proof of the rationality of the geometric Igusa-zeta series, we should mention that this method is different from working with classical arcs in the classical Grothendieck ring. Although we eventually take classes in $\mathbf{Gr}(F^{\text{form}})$, thus *collapsing nilpotents*, we will do this only after taking arcs. Put differently, although arcs will be reduced, the base schemes will not be, and to see that this makes a difference (even in regards to dimension!), we list, for small lengths, some defining equations of arcs and their reductions for three different closed subschemes with the same underlying set, the union of two lines in the plane:

TABLE 1. Equations and dimension d of arcs and their classes in $\mathbf{Gr}(F^{\text{form}})$

	l	$xy = 0$	d	$x^2y = 0$	d	$x^2y^3 = 0$	d
∇_l	1	$\tilde{x}_0\tilde{y}_0,$		$\tilde{x}_0^2\tilde{y}_0,$		$\tilde{x}_0^2\tilde{y}_0^3,$	
	2	$\tilde{x}_0\tilde{y}_1 + \tilde{x}_1\tilde{y}_0,$		$2\tilde{x}_0\tilde{x}_1\tilde{y}_0 + \tilde{x}_0^2\tilde{y}_1,$		$2\tilde{x}_0\tilde{x}_1\tilde{y}_0^3 + 3\tilde{x}_0^2\tilde{y}_0^2\tilde{y}_1,$	
	3	$\tilde{x}_0\tilde{y}_2 + \tilde{x}_1\tilde{y}_1 + \tilde{x}_2\tilde{y}_0$		$\tilde{x}_0^2\tilde{y}_2 + 2\tilde{x}_0\tilde{x}_1\tilde{y}_1 + (2\tilde{x}_0\tilde{x}_2 + \tilde{x}_1^2)\tilde{y}_0$		$3\tilde{x}_0^2(\tilde{y}_0\tilde{y}_1^2 + \tilde{y}_0^2\tilde{y}_2) + 6\tilde{x}_0\tilde{x}_1\tilde{y}_0^2\tilde{y}_1 + (\tilde{x}_1^2 + 2\tilde{x}_0\tilde{x}_2)\tilde{y}_0^3$	
$[\nabla_l]$	1	$\tilde{x}_0\tilde{y}_0,$	1	$\tilde{x}_0\tilde{y}_0,$	1	$\tilde{x}_0\tilde{y}_0,$	1
	2	$\tilde{x}_0\tilde{y}_1, \tilde{x}_1\tilde{y}_0,$	2	$\tilde{x}_0\tilde{y}_1,$	3	[no new equation]	3
	3	$\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_1, \tilde{x}_2\tilde{y}_0$	3	$\tilde{x}_0\tilde{y}_2, \tilde{x}_1\tilde{y}_0$	4	$\tilde{x}_1\tilde{y}_0$	5

Tagged and formal equations. In the sequel, we will only invert variables, and to this end, we simplify our notation even further. To any natural number a , we associate its *tagged* version a^* , and we call $v(a^*) := a$ the *underlying value* (or *untagged* version) of a^* . We can add tagged and/or untagged numbers by the rule that the underlying value of the sum is the sum of the underlying values of the terms, where the sum is tagged if and only if at least one term is tagged (e.g., $2 + 3^* = 5^*$). Fix $m \geq 1$, and let Γ_m be the collection of m -tuples with entries natural numbers or their tagged versions. We extend v component-wise to get a map $v: \Gamma_m \rightarrow \mathbb{N}^m$, sending a tuple $\theta \in \Gamma_m$ to its *underlying value* $v\theta$. We define a partial order on Γ_m by $\alpha \preceq \beta$ if and only if α_j is untagged and $\alpha_j \leq \beta_j$, or α_j is tagged and $\alpha_j = \beta_j$, for all $j = 1, \dots, m$.

We will introduce two equational conventions in this section that are useful for discussing arc equations. To each variable x , we associate its tagged version x_* , which we will treat as an invertible variable. Given a tagged number a^* , we let x^{a^*} be the same as $x_*^{a^*}$, and simply write x_*^a . Hence, we may associate to a polynomial $f \in F[x]$, the polynomial $f(x_*)$, which is just $f(x)$ but viewed in the Laurent polynomial ring $F[x, \frac{1}{x}]$. Therefore, we interpret the equation $f(x_*) = 0$ as the conjunction $f(x) = 0$ and x is invertible, that is to say, the pp-formula $(\exists x') f(x) = 0 \wedge xx' = 1$. We may extend this practice to several variables, tagging some of them and leaving the others unchanged. For instance, the *tagged* equation $x_*^2 + x_*y^3 + z_*^3 = 0$ should be considered as an element of the *mixed Laurent polynomial ring* $F[x, y, z, \frac{1}{x}, \frac{1}{z}]$, and is equivalent with the conditions $x^2 + xy^3 + z^3 = 0$ together with x and z are invertible.

Our second convention is the use of a formal variable ξ , fixed once and for all. Given a power series $f(x, \xi) \in F[x][[\xi]]$ (or, at times, a Laurent series) with coefficients in a polynomial ring $F[x]$ (or a mixed Laurent polynomial ring), we interpret the (formal) equation $f = 0$ as the condition on the x -variables that f be identical zero as a power series (Laurent series) in ξ . In other words, if $f(x, \xi) = f_0(x) + f_1(x)\xi + f_2(x)\xi^2 + \dots$, then $f = 0$ stands for the (infinite) conjunction $f_0 = f_1 = f_2 = \dots = 0$ (as $f = 0$ and $\xi^i f = 0$ yield equivalent systems of equations, we may reduce the Laurent series case to the power series case). Similarly, for each n , the equivalence $f(x, \xi) \equiv 0 \pmod{\xi^n}$ stands for the conjunction $f_0 = f_1 = \dots = f_{n-1} = 0$. An example of a combination of both conventions is

$$0 = (x + y_*\xi)^2 + (z_* + w\xi)^3$$

which is equivalent to the pp-formula

$$\begin{aligned} x^2 + z^3 &= 0 \\ 2xy + 3z^2w &= 0 \\ y^2 + 3zw^2 &= 0 \\ w^3 &= 0 \\ (\exists y', z') \quad yy' = 1 \wedge zz' = 1 \end{aligned}$$

To any m -tuple of variables $x = (x_1, \dots, x_m)$, we associate the corresponding (countably many) *arc variables* $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \dots)$, where each \tilde{x}_i is an m -tuple $(\tilde{x}_{i,1}, \dots, \tilde{x}_{i,m})$. For each i , we let

$$\dot{x}_i = \tilde{x}_{0,i} + \tilde{x}_{1,i}\xi + \tilde{x}_{2,i}\xi^2 + \dots$$

be the *generic arc series* in ξ , and we write \dot{x} for the tuple $(\dot{x}_1, \dots, \dot{x}_m)$. Given $\theta \in \Gamma_m$, we define $\dot{x}(\theta)$ to be the m -tuple of twisted power series with i -th entry equal to

$$\dot{x}_i(\theta) := \tilde{x}_{\theta_i,i} + \tilde{x}_{\theta_i+1,i}\xi + \tilde{x}_{\theta_i+2,i}\xi^2 + \dots$$

if θ_i is untagged, and

$$(\dot{x}_i)_*(\theta) := (\tilde{x}_{v\theta_i,i})_* + \tilde{x}_{v\theta_i+1,i}\xi + \tilde{x}_{v\theta_i+2,i}\xi^2 + \dots$$

if θ_i is tagged (note that according to this convention, only the constant term is actually tagged, which accords with the fact that a power series is invertible if and only if its constant term is). For each $\theta \in \Gamma_m$, define a change of variables τ_θ sending, for each $j = 1, \dots, m$, the variable $\tilde{x}_{i,j}$ to $\tilde{x}_{i-v\theta_j,j}$ and $(\tilde{x}_{i,j})_*$ to $(\tilde{x}_{i-v\theta_j,j})_*$ for $i \geq v\theta_j$, and leaving the remaining variables and their tagged versions unchanged. In particular, τ_θ only depends on the underlying value of θ , and $\tau_\theta(\dot{x}_i(\theta))$ is equal to \dot{x}_i if θ_i is untagged and to $(\dot{x}_i)_*$ if θ_i is tagged.

Directed arcs. With these conventions, we can now write down the equations of an arc scheme more succinctly. If $X \subseteq \mathbb{A}_F^m$ is the closed subscheme defined by the schemic formula $g_1 = \dots = g_s = 0$, then $\nabla_n X$ is defined by the conditions

$$g_1(\dot{x}) \equiv g_2(\dot{x}) \equiv \dots \equiv g_s(\dot{x}) \equiv 0 \pmod{\xi^n}$$

and $\dot{x}(\mathbf{n}) = 0$ (recall that \mathbf{n} is the tuple all of whose entries are equal to n). Note that the latter condition simply means that $\tilde{x}_{i,j} = 0$ for all $i \geq n$ and all $j = 1, \dots, m$.

We extend the notion of arc scheme, by considering certain (initial) linear subspaces of arc schemes. Given $\theta \in \Gamma_m$, we define the n -th *directed arc scheme along* θ , denoted $\nabla_n^\theta X$, as the locally closed subscheme of $\nabla_n X$ defined by the conditions $\tilde{x}_{i,j} = 0$ for $i < \theta_j$, and $\tilde{x}_{v\theta_j,j}$ is invertible if θ_j is tagged, for $j = 1, \dots, m$. We may also refer to $\nabla_n^\theta X$ as the subscheme of all *arcs along*, or *with initial direction* θ . Writing out these conditions in more detail, the defining equations of $\nabla_n^\theta X$ become

$$g_1(\dot{x}(\theta)) = \dots = g_s(\dot{x}(\theta)) \equiv 0 \pmod{\xi^n}$$

and $\tilde{x}_{i,j} = 0$ for $i < \theta_j$ or $i \geq n$, and for $j = 1, \dots, m$. By the change of variables τ_θ , we can rewrite these equations as

$$(40) \quad \tau_\theta(g_1(\dot{x}(\theta))) = \dots = \tau_\theta(g_s(\dot{x}(\theta))) \equiv 0 \pmod{\xi^n} \quad \wedge \quad \dot{x}(\mathbf{n} - \theta) = 0.$$

This form will be easier to work with, as we can now compare arcs along different directions; we call the first set of equations in (40) the *arc equations*, and the second set the *initial conditions*. We will use the arc equations as follows: given $\theta \in \Gamma_m$, let z be a new m -tuple of variables with corresponding arc variables \tilde{z} , called the θ -tagging of x , where z_j

is equal to $(x_j)_*$ or x_j depending on whether θ_j is tagged or not, and similarly, $\tilde{z}_{i,j} = \tilde{x}_{i,j}$ unless $i = 0$ and θ_j is tagged, in which case $\tilde{z}_{0,j} = (\tilde{x}_{0,j})_*$. If $f = \sum_{\mu} c_{\mu} x^{\mu}$, then

$$(41) \quad \tau_{\theta}(f(\dot{x}(\theta))) = \sum_{\mu} c_{\mu} \xi^{\mu\theta} \dot{z}^{\mu}.$$

9.4. Example (Fibers). Recall that $\rho_n: \nabla_n X \rightarrow X$ is the canonical projection of the arc scheme onto the base scheme. Let us calculate the fiber $\rho_n^{-1}(O)$ of the origin. If $f_1 = \dots = f_s = 0$ is the spheric formula defining X , then $\nabla_n X$ is given by the equations $f_i(\dot{x}) \equiv 0 \pmod{\xi^n}$, and $\rho_n^{-1}(O)$ is the closed subscheme given by $\tilde{x}_0 = 0$, that is to say,

$$(42) \quad \rho_n^{-1}(O) = \nabla_n^1 X.$$

9.5. Definition (Twisted geometric Igusa-zeta series). Given $\theta \in \Gamma_m$, define the θ -twisted geometric Igusa-zeta series of X to be

$$\text{Igu}_X^{\theta}(t) := \sum_{n=0}^{\infty} [\nabla_n^{\theta} X] t^n.$$

Hence, $\text{Igu}_X^{\text{geom}}$ is just the case in which the twist is zero.

At times, it is convenient, especially in inductive arguments, to prove that all twisted geometric Igusa-zeta series are rational.

Twisted initial forms. For the remainder of the section, we restrict to the case of a hypersurface X defined by a single equation $f := \sum_{\nu} c_{\nu} x^{\nu}$. If f is not homogeneous, we can no longer expect such a simple relation between the arc scheme and the fiber above the singular locus. As we will shortly see, the following hypersurfaces derived from X will play an important role: for every $\theta \in \Gamma_m$, let \tilde{X}^{θ} be defined as follows. View $F[x]$ as a graded ring giving the variable x_i weight $v\theta_i$. Let $\text{ord}_{\theta}(f)$, or $\text{ord}_{\theta}(X)$, be the order of f in this grading, that is to say, the minimum of all $v\theta \cdot \nu$ with $c_{\nu} \neq 0$, and let \tilde{X}^{θ} be the hypersurface with defining equation

$$\tilde{f}^{\theta} := \sum_{v\theta \cdot \nu = \text{ord}_{\theta}(X)} c_{\nu} x^{\nu}.$$

In particular, $X = \tilde{X}^0$. We call \tilde{X}^{θ} , or rather, \tilde{f}^{θ} , the θ -twisted initial form of X .

Here is an example to view the previous conventions and definitions at work:

9.6. Example. Let $f = x^9 + x^2 y^4 + z^4$ and $\theta = (2, 3^*, 5)$. Hence $\nabla_n^{(2,3^*,5)} X$ is the locally closed subscheme of $\nabla_{(2,3,5)} X$ given by the conditions $\tilde{x}_0 = \tilde{x}_1 = \tilde{y}_0 = \tilde{y}_1 = \tilde{y}_2 = \tilde{z}_0 = \tilde{z}_1 = \tilde{z}_2 = \tilde{z}_3 = \tilde{z}_4 = 0$ and \tilde{y}_3 is invertible. Using (40) and (41), its equations are

$$\tau_{(2,3^*,5)}(f(\dot{x}(2, 3^*, 5), \dot{y}(2, 3^*, 5), \dot{z}(2, 3^*, 5))) = \xi^{18} \dot{x}^9 + \xi^{16} \dot{x}^2 \dot{y}_*^4 + \xi^{20} \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

and $\dot{x}(n-2) = \dot{y}(n-3) = \dot{z}(n-5) = 0$. Hence, $\text{ord}_{(2,3^*,5)}(X) = 16$ and the twisted initial form $\tilde{X}^{(2,3^*,5)}$ is given by $\tilde{f}^{(2,3^*,5)} = x^2 y_*^4$, that is to say, by the two conditions $x^2 y^4 = 0$ and y is a unit.

Regular base. We will deduce rationality by splitting off regular pieces of various twisted initial forms, until we arrive at a recursive relation involving the arc scheme of the original hypersurface. To this end, we introduce the following definition: we say that $\theta \in \Gamma_m$ is X -regular if \tilde{X}^{θ} is smooth. As with arcs, directed arcs above a regular base have a locally trivial fibration, a fact which will allow us to determine their contribution to the Igusa-zeta series:

9.7. Proposition. *Let $X \subseteq \mathbb{A}^m$ be a hypersurface. For each X -regular tuple $\theta \in \Gamma_m$, we have locally (i.e., on an open affine covering), an isomorphism*

$$\nabla_n^\theta X \cong \nabla_{n-\text{ord}_X(\theta)}(\tilde{X}^\theta) \times \mathbb{A}^{m \cdot \text{ord}_X(\theta) - |\theta|},$$

for each $n > \text{ord}_X(\theta)$.

Proof. Let $f = \sum_\nu c_\nu x^\nu$ be the defining equation of X . Let us put $a := \text{ord}_X(\theta)$; recall that it is the minimum of all $v\theta \cdot \nu$ with $c_\nu \neq 0$. Instead of x , we will use the m -tuple of variables z whose i -th variable is tagged precisely if θ_i is. For each k , let $f_k^\theta := \sum_{v\theta \cdot \nu = k} c_\nu z^\nu$, so that $\tilde{f}^\theta = f_a^\theta$ is the defining equation of \tilde{X}^θ . By (40) and (41), the arc equation of $\nabla_n^\theta X$ is

$$\tau_\theta(f(\dot{z}(\theta))) = \sum_{k=a}^{n-1} \xi^k f_k^\theta(\dot{z}) \equiv 0 \pmod{\xi^n}$$

whereas the initial condition is $\dot{z}(\mathbf{n} - \theta) = 0$. Factoring out ξ^a , yields the arc equation

$$(43) \quad \sum_{k=0}^{n-a-1} \xi^k f_{a+k}^\theta(\dot{z}) \equiv 0 \pmod{\xi^{n-a}}.$$

On the other hand, the arc equation of $\nabla_{n-a} \tilde{X}^\theta$ is

$$(44) \quad \tilde{f}^\theta(\dot{z}) \equiv 0 \pmod{\xi^{n-a}}.$$

Note that expansions (43) and (44) have the same constant term $\tilde{f}^\theta = f_a^\theta$. Expand

$$f_k^\theta(\dot{z}) = \sum_l \xi^l f_{k,l}^\theta(\tilde{z}),$$

with each $f_{k,l}^\theta$ only depending on $\tilde{z}_0, \dots, \tilde{z}_{l-1}$. If $k = a$, we will write $\tilde{f}_l^\theta(\tilde{z})$ for $f_{a,l}^\theta(\tilde{z})$, so that $\tilde{f}^\theta(\dot{z}) = \sum_l \xi^l \tilde{f}_l^\theta(\tilde{z})$. Substituting in (43), we get an expansion

$$\sum_{k,l=0}^{n-a-1} \xi^{k+l} f_{a+k,l}^\theta(\tilde{z})$$

showing that the defining equations of $\nabla_n^\theta X$ are $g_0 = \dots = g_{n-a-1} = 0$ together with $\tilde{z}_{i,j} = 0$ if $i \geq n - \theta_j$, where

$$(45) \quad g_l(\tilde{z}) := \sum_{k=0}^l f_{a+k,l-k}^\theta = \tilde{f}_l^\theta + \sum_{k=1}^l f_{a+k,l-k}^\theta.$$

Since \tilde{X}^θ is smooth, the proof of Corollary 7.20 shows that locally \tilde{f}_l^θ , for $l > 0$, is linear in the \tilde{z}_l -variables, and smoothness allows us to solve for one of the \tilde{z}_l -variables in terms of the others. Restricting to a basic open, we may assume that we can do this globally (we leave the details to the reader; but see also the proof of Corollary 7.20). However, the same is then true for g_k , since the difference $g_l - \tilde{f}_l^\theta$ only depends on variables $\tilde{z}_0, \dots, \tilde{z}_{l-1}$ by (45). This shows that the closed subscheme defined by g_0, \dots, g_{n-a-1} when viewed in the variables $\tilde{z}_0, \dots, \tilde{z}_{n-a-1}$ is the same as $\nabla_n \tilde{X}^\theta$. As for the remaining ma variables $\tilde{z}_{n-a}, \dots, \tilde{z}_{n-1}$, among these, $|\theta|$ many of them are put equal to zero, whereas the rest remains free, proving the assertion. \square

9.8. Corollary. *If θ is X -regular, then*

$$[\nabla_n^\theta X] = [\tilde{X}^\theta] \cdot \mathbb{L}^{(m-1)(n-1) + \text{ord}_X(\theta) - |\theta|}$$

in $\mathbf{Gr}(F^{\text{pp}})$.

Proof. This follows, by the same argument as for Lemma 7.21, immediately from Corollary 7.20 and Proposition 9.7, noting that \tilde{X}^θ has dimension $m - 1$. \square

Recursion. Given $\alpha, \beta \in \Gamma_m$, we will write $\alpha \triangleleft_X \beta$, if $\alpha \preceq \beta$ and there exists some $s > 0$ such that

$$\tau_\beta(f(\dot{z}(\beta))) = \xi^s \tau_\alpha(f(\dot{z}(\alpha))).$$

An easy calculation shows that necessarily $s = \text{ord}_X(\beta) - \text{ord}_X(\alpha)$. Note that f is homogeneous in the classical sense if and only if $\mathbf{0} \triangleleft_X \mathbf{1}$.

9.9. Lemma. *If $\alpha \triangleleft_X \beta$, then*

$$[\nabla_n^\beta X] = [\nabla_{n-s}^\alpha X] \cdot \mathbb{L}^{sm - |\beta| + |\alpha|}$$

in $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$, for all $n > s$, with $s = \text{ord}_X(\beta) - \text{ord}_X(\alpha)$.

Proof. By (40), the defining equations of $\nabla_n^\beta X$ are

$$\tau_\beta(f(\dot{z}(\beta))) \equiv 0 \pmod{\xi^n} \quad \text{and} \quad \dot{x}(\mathbf{n} - \beta) = 0.$$

By assumption, the power series in the arc equation equals $\xi^s \tau_\alpha(f(\dot{z}(\alpha)))$, and so yields the arc equation

$$(46) \quad \tau_\alpha(f(\dot{z}(\alpha))) \equiv 0 \pmod{\xi^{n-s}}.$$

However, (46) is also the arc equation of $\nabla_{n-s}^\alpha X$. As the initial condition for $\nabla_{n-s}^\alpha X$ is given by $\dot{x}(\mathbf{n} - \mathbf{s} - \alpha)$, the difference between the two directed arc schemes lies in the number of free variables not covered by the respective initial conditions, a number which is easily seen to be $|\mathbf{s} - \beta + \alpha| = sm - |\beta| + |\alpha|$, whence the assertion. \square

Rationalizing trees. We are interested in subtrees of Γ_m , and will use the following terminology: by a *tree* we mean a finite, connected partially ordered subset of (nodes from) Γ_m such that any initial segment is totally ordered. The unique minimum is called the *root* of the tree, and any maximal element is called a *leaf*. By a *branch*, we will mean a chain $[\alpha, \beta]$ from a node α to a leaf β .

Let $\delta \leq \eta$ be binary vectors, that is to say, tuples with entries 0 or 1. We define a transformation e_δ^η on Γ_m as follows. For each i , let e_i simply be addition with the basis vector e_i on Γ_m (note that per our addition convention, each entry stays in whichever state, tagged or untagged, it was). On the other hand, we let e_i^* be the transformation which tags the i -th entry but leaves the remaining entries unchanged. Given a binary vector ε , we let e_ε (respectively, e_ε^*) be the composition of all e_i (respectively, all e_i^*) for which $\varepsilon_i = 1$. Note that all these transformations commute with each other. Finally, we let e_δ^η be the composition of e_δ and $e_{\eta-\delta}^*$. For instance,

$$\begin{aligned} e_{(0,0,0,1,0)}^{(1,1,0,1,0)}(2, 3^*, 1, 4, 1) &= e_{(1,1,0,0,0)}^* e_{(0,0,0,1,0)}(2, 3^*, 1, 4, 1) = \\ &= e_1^* e_2^* e_4(2, 3^*, 1, 4, 1) = (2^*, 3^*, 1, 5, 1). \end{aligned}$$

Note that $e_\delta^\eta(\theta)$ has underlying value equal to $v\theta + \delta$. More precisely, taking in account our addition convention, we have

$$e_\delta^\eta(\theta) = e_{\eta-\delta}^*(\theta + \delta).$$

Note that e_δ^η can fail to be an increasing function (if in the above example we replace $(0, 0, 0, 1, 0)$ by $(0, 1, 0, 0, 1, 0)$ the resulting tuple is $(2^*, 4^*, 1, 5, 1)$ which is not comparable with $(2, 3^*, 1, 4, 1)$ because of the second entry). A necessary condition is

$$(47) \quad (\forall i)[\text{if } \theta_i \text{ tagged then } \eta_i = 0] \Rightarrow \theta \preceq e_\delta^\eta(\theta).$$

We will use these transformation mainly through the following result:

9.10. Lemma. *Let $X \subseteq \mathbb{A}_F^m$ be a closed subscheme. For every $\theta \in \Gamma_m$, and every binary vector η , we have an identity*

$$[\nabla_n^\theta X] = \sum_{\delta \leq \eta} [\nabla_n^{e_\delta^\eta(\theta)} X]$$

in $\mathbf{Gr}(F^{\text{form}})$, for all n .

Proof. We apply (39) to $Y := \nabla_n^\theta X$ with respect to the variables $\tilde{x}_{\theta_i, i}$ such that $\eta_i = 1$, yielding

$$[Y] = \sum_{\delta \leq \eta} [\bar{Y}_\delta].$$

However, \bar{Y}_δ is obtained by inverting, that is to say, tagging $\tilde{x}_{\theta_i, i}$ if $\delta_i = 1$, and equating it to zero, if $\delta_i = 0$. Since the defining arc equations for $\nabla_n^\theta X$ are $f_i(\dot{x}(\theta)) \equiv 0 \pmod{\xi^n}$, for $i = 1, \dots, s$, where $f_1 = \dots = f_s = 0$ is the defining scheme formula of X , the arc equations of \bar{Y}_δ are $f_i(\dot{x}(\theta + \eta - \delta)) \equiv 0 \pmod{\xi^n}$, for $i = 1, \dots, s$, together with inverting all $\tilde{x}_{\theta_i, i}$ for which $\delta_i = 1$. As these are the defining arc equations for $\nabla_n^{e_{\eta-\delta}^\eta(\theta)} X$, we proved the assertion (note that summing over all δ is the same as summing over all $\eta - \delta$). \square

We define by induction on the height of a tree in Γ_m for it to be a *resolution tree* as follows: any singleton is a resolution tree; if T is a resolution tree, then so is T' which is obtained from T first by choosing a leaf γ of T and a binary vector η such that whenever an entry γ_i is tagged, the corresponding entry η_i is zero, and then by adding on to T all the $e_\delta^\eta(\gamma)$ as new leafs, for $\delta \leq \eta$. By (47), the new subset is indeed a tree. In particular, if every entry of some node $\theta \in T$ is tagged and T is a resolution tree, then θ is necessarily a leaf of T . Moreover, if T' is a subtree of T , that is to say, all nodes of T greater than or equal to a fixed node, and T is a resolution tree, then so is T' .

9.11. Lemma. *Let $X \subseteq \mathbb{A}^m$ be a closed subscheme and let $T \subseteq \Gamma_m$ be a subtree with root θ . If T is a resolution tree, then*

$$[\nabla_n^\theta X] = \sum_{\gamma \in T \text{ leaf}} [\nabla_n^\gamma X]$$

in $\mathbf{Gr}(F^{\text{form}})$, for all n .

Proof. By induction on the height of a node, immediate from Lemma 9.10. \square

For a tree $T \subseteq \Gamma_m$, we define recursively what it means for it to be *X-rationalizing*: if all but one of its leafs γ are *X-regular* and if $\theta \triangleleft_X \gamma$, then T is *X-rationalizing*. Furthermore, if T is *X-rationalizing*, γ a leaf of T , and T' an *X-rationalizing* tree with root γ , then the composite tree obtained by replacing γ in T by T' is again *X-rationalizing*.

9.12. Theorem. *If T is an X-rationalizing resolution tree with root $\theta \in \Gamma_m$, then the geometric Igusa-zeta series $\text{Igu}_X^{\text{geom}}$ is rational over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$.*

Proof. Let us first show that if T is a resolution tree with root θ and for each leaf γ , the twisted geometric Igusa-zeta series Igu_X^γ is rational, then so is Igu_X^θ . Indeed, by Lemma 9.11, we have an identity

$$(48) \quad [\nabla_n^\theta X] = \sum_{\gamma \in T \text{ leaf}} [\nabla_n^\gamma X]$$

Multiplying with t^n , and summing over all n then yields

$$\text{Igu}_X^\theta = \sum_{\gamma \in T \text{ leaf}} \text{Igu}_X^\gamma,$$

proving the claim. Hence, we may use the recursive definition of a rationalizing tree and the previous result, to reduce to the case that all but one leaf $\bar{\gamma}$ of T is X -regular, and $\theta \triangleleft_X \bar{\gamma}$. Assume first the characteristic is zero. Again (48) holds, and by Corollary 9.8, the directed arcs along all leafs $\gamma \neq \bar{\gamma}$ are certain multiples of $\mathbb{L}^{(m-1)n}$, where the multiple is independent from n , whereas by Lemma 9.9, the directed arc class along $\bar{\gamma}$ is a multiple of the class along θ . More precisely, there exists an element $w \in \mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$ such that

$$[\nabla_n^\theta X] = w \cdot \mathbb{L}^{(m-1)n} + [\nabla_{n-s}^\theta X] \cdot \mathbb{L}^r$$

for all n , where $s = \text{ord}_X(\bar{\gamma}) - \text{ord}_X(\theta)$ and $r = sm - |\bar{\gamma}| + |\theta|$. Multiplying with t^n and summing over all sufficiently large n , we get an identity

$$\text{Igu}_X^\theta = \frac{Q}{1 - \mathbb{L}^{m-1}t} + \mathbb{L}^r t^s \text{Igu}_X^\theta$$

for some polynomial Q over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$. Solving for Igu_X^θ then proves the claim, as $s > 0$. \square

Linear rationalization algorithm. The algorithm that we will use here to construct an X -rationalizing resolution tree with root $\mathbf{0}$, thus establishing the rationality of the geometric Igusa-zeta series of a hypersurface X by Theorem 9.12, relies on the simple form the singular locus takes. Namely, we say that a hypersurface X containing the origin has *linear singularities*, if its singular locus is a union of coordinate subspaces, where a *coordinate subspace* is a closed subscheme given by equations $x_{i_1} = \cdots = x_{i_s} = 0$ for some subset x_{i_j} of the variables. We will apply the algorithm to hypersurfaces all of whose twisted initial forms have linear singularities.

Single-branch linear rationalization algorithm for power hypersurfaces with an isolated singularity. In its simplest form, the algorithm works as follows: assume for every twisted initial form \tilde{X}^θ of X , there exists a variable x_i such that the basic subset $(\tilde{X}^\theta)_{x_i}$ is smooth. We then apply Lemma 9.10 with $\eta = e_i$, thus building a binary tree with at each stage exactly one untagged and one tagged leaf, and such that the latter is moreover X -regular, whence requires no further action. We continue this process (on the remaining untagged leaf) until we reach an untagged leaf γ with $\mathbf{0} \triangleleft_X \gamma$, at which point we can invoke Theorem 9.12. If such a leaf γ can be found, we say that the algorithm *stops*.

This algorithm will stop on any hypersurface X with an equation of the form

$$f := r_1 x_1^{a_1} + \cdots + r_m x_m^{a_m}$$

with $a_i > 0$ and $r_i \in F$; we will refer to such an X as *power hypersurface*. In characteristic zero, the origin is an isolated singularity, but in positive characteristic, this is only the case if at most one of the powers a_i is divisible by the characteristic. In the isolated singularity case, the algorithm as described above does apply: any twisted initial form is again a power hypersurface; if it is one of the powers $x_i^{a_i}$, its regular locus, although empty, is obtained

by inverting x_i , even if a_i is divisible by the characteristic; in the remaining case, we can always invert one variable whose power is not divisible by the characteristic, yielding a smooth twisted initial form. So remains to show that this algorithm stops, that is to say, will eventually produce a leaf γ such that $\mathbf{0} \triangleleft_X \gamma$. To see this, note that the set of all $\text{ord}_X(\theta)$, with θ running over all untagged nodes in the tree, is equal to the union of all semi-groups $a_i\mathbb{N}$, for $i = 1, \dots, m$. Therefore, if e is the least common multiple of all a_i , it will occur as some $\text{ord}_X(\gamma)$ for some untagged leaf γ in this algorithm. However, it is easy to see that $\tilde{X}^\gamma = X$, and hence we showed:

9.13. Theorem. *The geometric Igusa zeta-series $\text{Igu}_X^{\text{geom}}$ of a power hypersurface X with an isolated singularity is rational over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$. \square*

In the next section, we will work out in complete detail the implementation of this algorithm for the power surface $x^2 + y^3 + z^4 = 0$. Generalizing these calculations, we will derive the following formula:

9.14. Corollary. *If $r_1x_1^{a_1} + \dots + r_mx_m^{a_m} = 0$ is the equation of the power hypersurface X with an isolated singularity, then there exists a polynomial $Q_X(t) \in \mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}[t]$ such that*

$$\text{Igu}_X^{\text{geom}} = \frac{Q_X(t)}{(1 - \mathbb{L}^{m-1}t)(1 - \mathbb{L}^N t^e)}$$

where e is the least common multiple of a_1, \dots, a_m , and where

$$(49) \quad N = e\left(\frac{a_1 - 1}{a_1} + \dots + \frac{a_m - 1}{a_m}\right).$$

10. RATIONALITY OF THE IGUSA-ZETA SERIES FOR DU VAL SURFACES

In this final section, we apply the previous rationalization algorithm to the geometric Igusa zeta series of Du Val surfaces, which over a field F of characteristic different $p \neq 2$, are precisely the isolated canonical singularities (at the origin O). Over \mathbb{C} , they can be realized, up to analytic isomorphism, as the quotients \mathbb{A}^2/Γ , where $\Gamma \subseteq \text{SL}_2(\mathbb{C})$ is a finite subgroup. A complete invariant is the dual resolution graph viewed as one of the following Dynkin diagrams: A_k , D_k , E_6 , E_7 , or E_8 , and we therefore will denote them simply by the latter letters. The main result of this section is the rationality of their geometric Igusa-zeta series over $\mathbf{Gr}(F^{\text{form}})_{\mathbb{L}}$, summarized by the following table, where we listed in the last column only the relevant factor in the denominator (the other factor being $(1 - \mathbb{L}^2t)$).

TABLE 2. Denominator of the Igusa-zeta series for Du Val surfaces ($p \neq 2$).

Du Val surface X		equation	denominator of $(1 - \mathbb{L}^2t) \text{Igu}_X^{\text{geom}}$
A_k	k odd	$x^2 + y^2 + z^{k+1}$	$(1 - \mathbb{L}^{2k+1}t^{k+1})$
	k even		$(1 - \mathbb{L}^{4k+2}t^{2k+2})$
D_k		$x^2 + y^2z + z^{k-1}$	$(1 - \mathbb{L}^{4k-5}t^{2k-2})$
E_6		$x^2 + y^3 + z^4$	$(1 - \mathbb{L}^{23}t^{12})$
E_7		$x^2 + y^3 + yz^3$	$(1 - \mathbb{L}^{29}t^{18})$
E_8		$x^2 + y^3 + z^5$	$(1 - \mathbb{L}^{60}t^{30})$

The E_6 -surface. Let us work step-by-step through the rationalization algorithm for the Du Val surface E_6 with equation $x^2 + y^3 + z^4$. We take a ‘short-cut’ by observing that the origin O is an isolated singularity, so that we only need to calculate the class of $\nabla_n^{(1,1,1)} E_6 = \rho_n^{-1}(O)$ by Theorem 9.3. By (41), its arc equations are

$$\xi^2 \dot{x}^2 + \xi^3 \dot{y}^3 + \xi^4 \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

together with the initial conditions $\tilde{x}_i = \tilde{y}_i = \tilde{z}_i = 0$ for $i \geq n-1$. The twisted initial form is x^2 . According to the algorithm, we have a single branching given by the transformations e_1^* and e_1 . The twisted initial form of $e_1^*(1, 1, 1) = (1^*, 1, 1)$ is defined by $x_*^2 = 0$ and hence is empty. So remains the untagged leaf $e_1(1, 1, 1) = (2, 1, 1)$, with arc equations

$$\xi^4 \dot{x}^2 + \xi^3 \dot{y}^3 + \xi^4 \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

and in addition to the previous initial conditions, also $\tilde{x}_{n-2} = 0$. As the twisted initial form is y^3 , we branch with e_2^* and e_2 . The twisted initial form of $e_2^*(2, 1, 1) = (2, 1^*, 1)$ is $y_*^3 = 0$, whence empty, leaving us with $e_2(2, 1, 1) = (2, 2, 1)$, whose arc equations are

$$(50) \quad \xi^4 \dot{x}^2 + \xi^6 \dot{y}^3 + \xi^4 \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

and an additional initial condition $\tilde{y}_{n-2} = 0$. The new twisted initial form is $x^2 + z^4$. At this point, inverting either variable x or z yields a regular surface. However, instead of choosing one, we may perform a multi-branching step, in which we consider all four possibilities $e_1 e_3$, $e_1^* e_3$, $e_1 e_3^*$, or $e_1^* e_3^*$, when applying (39), yielding the four leafs $(3, 2, 2)$, $(2^*, 2, 2)$, $(3, 2, 1^*)$, and $(2^*, 2, 1^*)$ respectively. The corresponding initial forms are given by $x^2 + y^3 = 0$, $x_*^2 = 0$, $z_*^4 = 0$, and $x_*^2 + z_*^4 = 0$. The middle two clearly are empty, and as the last is smooth, we may invoke Corollary 9.8, to get

$$\left[\nabla_n^{(2^*, 2, 1^*)} E_6 \right] = \left[\tilde{E}_6^{(2^*, 2, 1^*)} \right] \cdot \mathbb{L}^{2n-2+4-5} = [x_*^2 + z_*^4] \cdot \mathbb{L}^{2n-3}$$

as $\text{ord}_{(2^*, 2, 1^*)}(E_6) = 4$. This leaves the first leaf, $(3, 2, 2)$, with arc equations

$$\xi^6 \dot{x}^2 + \xi^6 \dot{y}^3 + \xi^8 \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

and the two additional initial conditions $\tilde{x}_{n-3} = \tilde{z}_{n-2} = 0$. Its twisted initial form $x^2 + y^3$ becomes non-singular if we invert x or y , suggesting another multi-branching step. Inverting one and equating the other to zero leads once more to contradictory equations, so we only have to deal with the two leafs $(3^*, 2^*, 2)$ and $(4, 3, 2)$. For the former, we may invoke once more Corollary 9.8, yielding the class

$$[x_*^2 + y_*^3 = 0] \cdot \mathbb{L}^{2n-2+6-7},$$

as $\text{ord}_{(3^*, 2^*, 2)}(E_6) = 6$. The latter has arc equations

$$\xi^8 \dot{x}^2 + \xi^9 \dot{y}^3 + \xi^8 \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

together with the vanishing of \tilde{x}_i , \tilde{y}_i and \tilde{z}_i for i greater than or equal to respectively $n-4$, $n-3$, and $n-2$. Since $(4, 3, 2)$ has the same twisted initial form as $(2, 2, 1)$, we may repeat our previous argument. Tagging both variables gives the leaf $(4^*, 3, 2^*)$ and

$$[x_*^2 + z_*^4] \cdot \mathbb{L}^{2n-2+8-9}$$

as $\text{ord}_{(4^*, 3, 2^*)}(E_6) = 8$. The latter leaf is $(5, 3, 3)$, with arc equations

$$\xi^{10} \dot{x}^2 + \xi^9 \dot{y}^3 + \xi^{12} \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

together with $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i = 0$ for $i \geq n-5$, $n-3$, $n-3$ respectively. As the twisted initial form is y^3 , we again branch over e_2^* and e_2 leading to the leaf $(5, 4, 3)$, with arc equations

$$\xi^{10} \dot{x}^2 + \xi^{12} \dot{y}^3 + \xi^{12} \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

together with $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i = 0$, for $i \geq n-5, n-4, n-3$ respectively. As x^2 is the new twisted initial form, we branch over e_1^* and e_1 , yielding the leaf $(6, 4, 3)$, with arc equations

$$\xi^{12} \dot{x}^2 + \xi^{12} \dot{y}^3 + \xi^{12} \dot{z}^4 \equiv 0 \pmod{\xi^n}$$

together with $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i = 0$ for $i \geq n-6, n-4, n-3$ respectively. As X itself is the twisted initial form of this leaf, that is to say, $\mathbf{0} \triangleleft_X (6, 4, 3)$, our algorithm has come to a halt. Indeed, if we factor out ξ^{12} in the last equation, we get the $(n-12)$ -th arc equations. Since we have $|(6, 4, 3)| = 13$ additional initial conditions, we are left with $3 \cdot 12 - 13 = 23$ free variables $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i$ for $n-12 \leq i < n-6, n-4, n-3$ respectively, as predicted by Lemma 9.9. Putting everything together, we showed that $[\nabla_n E_6]$ is equal to

$$(51) \quad [E_6 - O] \mathbb{L}^{2n-2} + 2 \left[\tilde{E}_6^{(2^*, 2, 1^*)} \right] \mathbb{L}^{2n-3} + \left[\tilde{E}_6^{(3^*, 2^*, 2)} \right] \mathbb{L}^{2n-3} + [\nabla_{n-12} E_6] \mathbb{L}^{23}$$

Multiplying with t^n , summing over all n , and solving for the zeta series yields

$$\mathrm{Igu}_{E_6}^{\mathrm{geom}} = \frac{Q_{E_6}}{(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{23} t^{12})}$$

for some polynomial Q_{E_6} over $\mathbf{Gr}(F^{\mathrm{form}})_{\mathbb{L}}$. A schematic representation of these calculations is given by the following rationalization tree, in which we equated, for brevity, a leaf to the class of the corresponding directed arc scheme (giving only its defining polynomial):

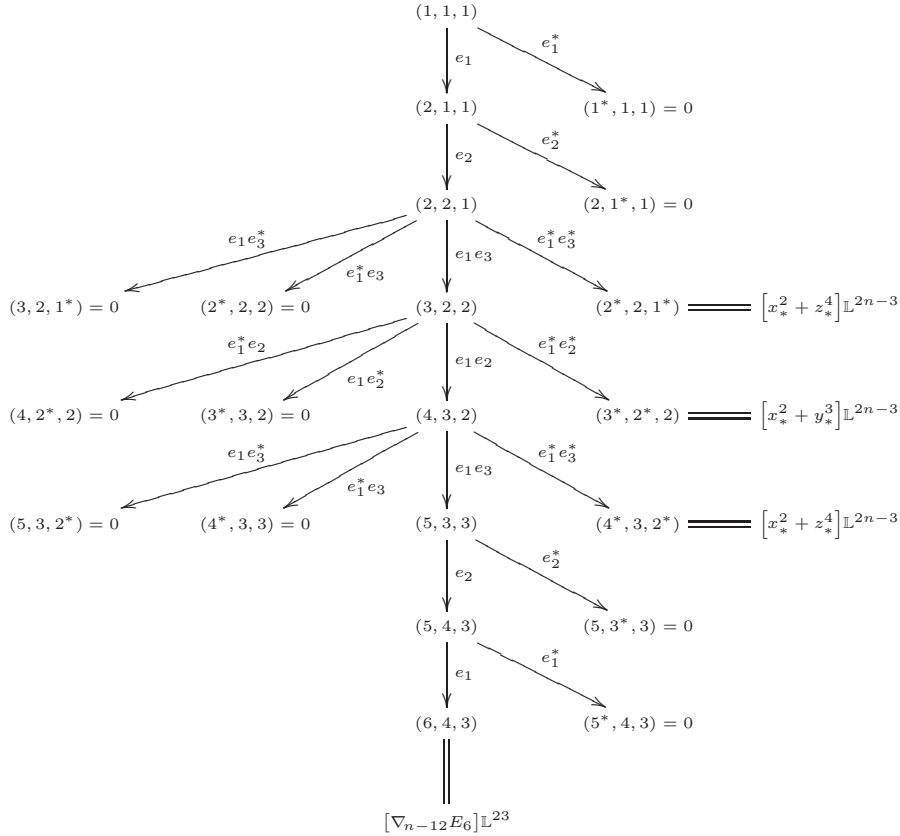


TABLE 3. The rationalization tree for the E_6 -surface

It is now also clear how this generalizes to any power hypersurface, yielding a proof of Corollary 9.14. Indeed, with e the least common multiple of the a_i , the algorithm stops at the leaf $\gamma := (\frac{e}{a_1}, \dots, \frac{e}{a_m})$, whose order is e . During this process, we introduced $|\gamma|$ many additional initial conditions. As we have me more arc variables for the n -th arc as for the $(n - e)$ -th arc, this yields $N = me - |\gamma|$ free variables, explaining formula (49).

Let us apply this algorithm also to the Du Val surfaces A_k and E_8 . The former is given by $x^2 + y^2 + z^{k+1}$. If k is odd, then $e = k + 1$ and $N = 3(k + 1) - (\frac{k+1}{2} + \frac{k+1}{2} + 1) = 2k + 1$, and hence

$$\text{Igu}_{A_k}^{\text{geom}} = \frac{Q_{A_k}}{(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{2k+1} t^{k+1})}.$$

If k is even, then $e = 2(k + 1)$ and $N = 6(k + 1) - (k + 1 + k + 1 + 2) = 4k + 2$, so that

$$\text{Igu}_{A_k}^{\text{geom}} = \frac{Q_{A_k}}{(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{4k+2} t^{2(k+1)})}.$$

Finally, since E_8 has equation $x^2 + y^3 + z^5$, the values are $e = 30$ and $N = 90 - (15 + 10 + 5) = 60$, so that

$$\text{Igu}_{E_8}^{\text{geom}} = \frac{Q_{E_8}}{(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{60} t^{30})}.$$

Although a priori \tilde{X}^θ depends on the embedding of X in some affine space, its class may be more independent from this embedding. For instance, in (51), all but the two middle terms are independent from an embedding. To which extent does this hold?

The E_7 -surface. This time, the defining equation is $x^2 + y^3 + yz^3 = 0$ (and, as before, assuming that the characteristic is not equal to 2 or 3), which again has an isolated (canonical) singularity. As this is no longer just a sum of powers, it will lead to a more complicated rationalization tree, given in Table 4 below.

Since in any rationalization tree, the sum of all nodes is equal to the root, we get, using Theorem 9.3, that

$$[\nabla_n E_7] = q \mathbb{L}^{2n-3} + [\nabla_{n-18} E_7] \mathbb{L}^{29}$$

where q is equal to

$$[E_7 - O] \mathbb{L} + [x^2 + yz_*^3] + [yz_*^3] \mathbb{L}^{-1} + 2[x_*^2 + y_*^3] + [x_*^2 + yz_*^3] + [y_*^3 + y_* z_*^3].$$

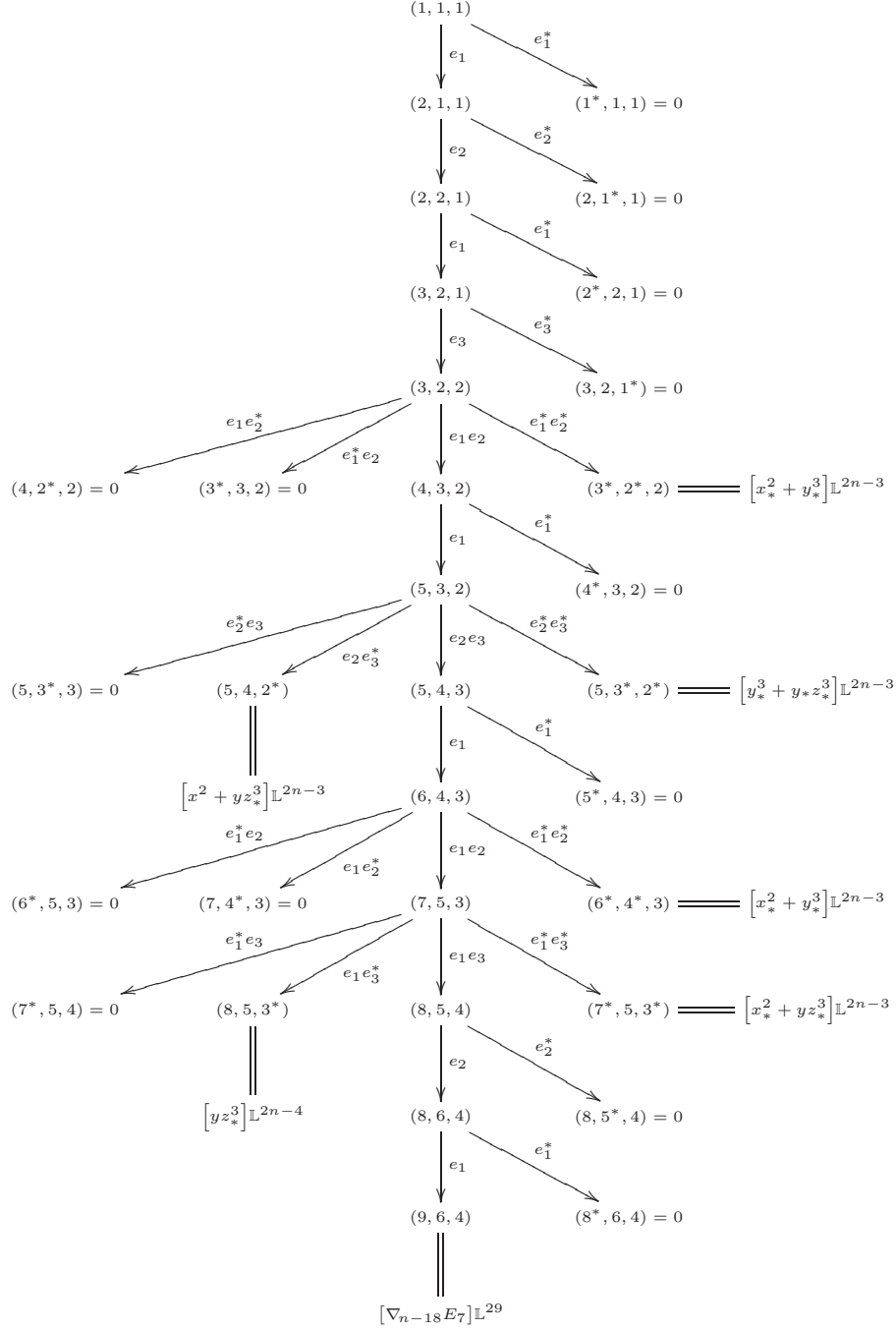
Using the identities

$$\begin{aligned} [x^2 + yz_*^3] &= \mathbb{L}^2 - \mathbb{L} = [yz_*^3] \\ [x_*^2 + yz_*^3] &= (\mathbb{L} - 1)^2 \\ [y_*^3 + y_* z_*^3] &= (\mathbb{L} - 1)[x_*^2 + y_*^3] \end{aligned}$$

we get

$$q = \mathbb{L}[E_7 - O] + (\mathbb{L} + 1)[x_*^2 + y_*^3] + 2\mathbb{L}(\mathbb{L} - 1).$$

Regardless the value of q , the usual argument yields the rationality of $\text{Igu}_{E_7}^{\text{geom}}$, with denominator equal to $(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{29} t^{18})$.

TABLE 4. The rationalization tree for the E_7 -surface

The D_k -surface. The general equation of the D_k -surface is $x^2 + y^2z + z^{k-1}$ for $k \geq 4$ (assuming the characteristic is different from 2). Depending on whether k is odd or even, we have two slightly different rationalization trees, both leading from the root $(1, 1, 1)$ to $(k-1, k-2, 2)$, where

$$\left[\nabla_n^{(k-1, k-2, 2)} D_k \right] = [\nabla_{n-2k+2} D_k] \cdot \mathbb{L}^{4k-5}$$

since we have $3(2k-2) - |(k-1, k-2, 2)| = 4k-5$ free variables. The equations for the directed arc along the starting value $(1, 1, 1)$ has arc equations

$$\xi^2 \dot{x}^2 + \xi^3 \dot{y}^2 \dot{z} + \xi^{k-1} \dot{z}^{k-1} \equiv 0 \pmod{\xi^n}.$$

It is not hard to see that the initial part of the tree is given by alternating e_1 and e_2 . As a result, in respectively the first and second term, the power of ξ is each time increased by 2. This goes on until one of them catches up with the power ξ^{k-1} , and this depends on the parity of k . So assume first that k is odd. In that case, we arrive at a leaf with value $(\frac{k-1}{2}, \frac{k-1}{2}, 1)$, whose directed arc has arc equations

$$\xi^{k-1} \dot{x}^2 + \xi^k \dot{y}^2 \dot{z} + \xi^{k-1} \dot{z}^{k-1} \equiv 0 \pmod{\xi^n}.$$

The remainder of the tree is now as follows, where the middle part gets repeated until the indicated value is reached:

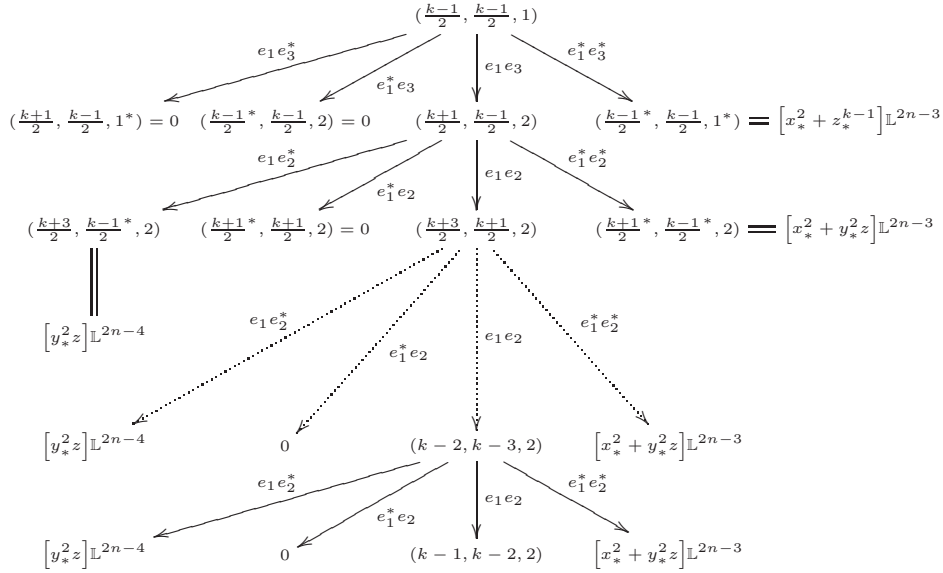


TABLE 5. Bottom part of the rationalization tree for D_k , when k is odd.

The tree for even k is analogous, where this time, the starting value is $(\frac{k}{2}, \frac{k-2}{2}, 1)$, with arc equations

$$\xi^k \dot{x}^2 + \xi^{k-1} \dot{y}^2 \dot{z} + \xi^{k-1} \dot{z}^{k-1} \equiv 0 \pmod{\xi^n}.$$

The remainder of the tree, with the middle part again repeated, is

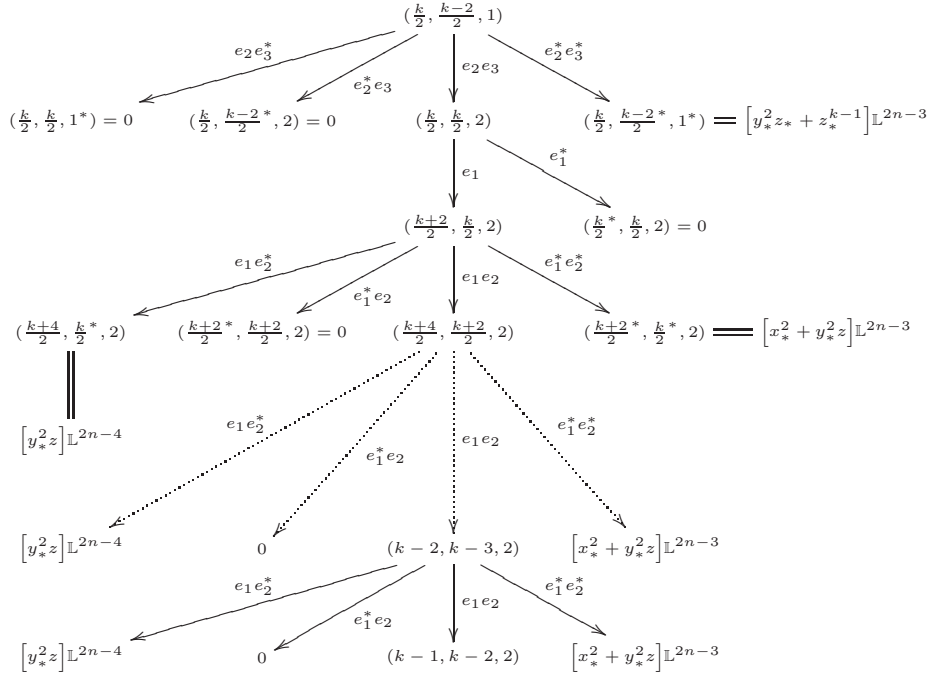


TABLE 6. Bottom part of the rationalization tree for D_k , when k is even.

Note that $[y_*^2 z] = \mathbb{L}^2 - \mathbb{L}$ appears $\lceil \frac{k}{2} \rceil$ times as an end value, and $[x_*^2 + y_*^2 z] = (\mathbb{L} - 1)^2$ appears $\lceil \frac{k+2}{2} \rceil$ many times. It follows that $[\nabla_n D_k]$ is equal to

$$([D_k - O]\mathbb{L} + [x_*^2 + z_*^{k-1}] + \frac{k+1}{2}(\mathbb{L}^2 - \mathbb{L}) + (\mathbb{L} - 1)^2)\mathbb{L}^{2n-3} + [\nabla_{n-2k+2} D_k]\mathbb{L}^{4k-5}$$

in the odd case, and to

$$([D_k - O]\mathbb{L} + [y_*^2 z_* + z_*^{k-1}] + \frac{k}{2}(\mathbb{L}^2 - \mathbb{L}) + (\mathbb{L} - 1)^2)\mathbb{L}^{2n-3} + [\nabla_{n-2k+2} D_k]\mathbb{L}^{4k-5}$$

in the even case. In particular, the geometric Igusa-zeta series $\text{Igu}_{D_k}^{\text{geom}}$ is rational with denominator equal to $(1 - \mathbb{L}^2 t)(1 - \mathbb{L}^{4k-5} t^{2k-2})$.

11. APPENDIX: IDEMPOTENTS IN NOETHERIAN LOCAL RINGS

Let us call two elements in a ring A *orthogonal* if their product is zero. Recall that an element $e \in A$ is called *idempotent*, if $e^2 = e$; the set of all idempotents in A will be denoted $\text{Idem}(A)$. If e is idempotent, then so is $1 - e$. Moreover, e and $1 - e$ are orthogonal, and the ideals they generate are subrings of A with the property that $A \cong eA \oplus (1 - e)A$. In fact, $A/eA \cong (1 - e)A$. In particular, A is indecomposable, or, equivalently, $\text{Spec } A$ is connected, if and only if 0 and 1 are its only idempotents. We define a partial order relation on $\text{Idem}(A)$ by $e \leq f$ if and only if $eA \subseteq fA$. Clearly, $0 \leq e \leq 1$, for all $e \in \text{Idem}(A)$. An idempotent is called *primitive* if it cannot be written as the sum of two non-zero orthogonal idempotents. Note that the sum of two orthogonal idempotents is again idempotent.

11.1. Lemma. *For idempotents $e, f, g \in A$, we have*

- (1) $e \leq f$ if and only if $e = ef$;
- (2) if $e = f + g$ with f and g orthogonal idempotents, then $f, g \leq e$
- (3) 0 is primitive;
- (4) a non-zero idempotent is primitive if and only if it cannot be written as the sum of two strictly smaller orthogonal idempotents;
- (5) a non-zero idempotent is primitive if and only if it is minimal among all non-zero idempotents.

Proof. One direction in (1) is clear, so assume $e \in fA$, say, $e = af$. Squaring this equality yields $e = a^2f = a(af) = ae$. On the other hand, $0 = (1 - e)e = (1 - e)af$, so that $af = eaf = (ae)f = ef$, and hence $e = ef$. For (2), multiply with f to get $ef = f + fg = f$, whence $f \leq e$ by (1). To prove (3), suppose $e + f = 0$ with e, f orthogonal idempotents. By (2), we get $e, f \leq 0$, and hence $e = f = 0$. Property (4) follows immediately from (2). For (5), let e be primitive and suppose $0 \neq f \leq e$, whence $f = fe$ by (1). Multiplying $1 = f + (1 - f)$ with e yields $e = ef + e(1 - f)$, so that by primitivity one of the two terms must be zero. As $f = ef \neq 0$, we get $e(1 - f) = 0$, whence $e = ef = f$. Conversely, assume e is minimal among all non-zero idempotents, but assume it is not primitive. Hence, by (4), we can write it as $e = f + g$ with $f, g < e$ orthogonal idempotents. By minimality, $f = g = 0$, contradicting that $e \neq 0$. \square

11.2. Lemma. *If A is Noetherian, then the order relation on $\text{Idem}(A)$ satisfies the ascending and descending chain conditions.*

Proof. Since A is Noetherian, ideal inclusion satisfies the ascending chain condition, and, therefore, so does \leq . To prove that \leq also satisfies the descending chain condition, observe that $e \leq f$ is equivalent with $1 - f \leq 1 - e$ by (1), so that we may apply the ascending chain condition to the latter. \square

11.3. Proposition. *If A is Noetherian, then there are only finitely many primitive idempotents, say, e_1, \dots, e_s . The e_i are mutually orthogonal, and any idempotent e can be written in a unique way as a sum $e = e_{i_1} + \dots + e_{i_t}$ with $1 \leq i_1 < \dots < i_t \leq s$. In particular, $e_1 + \dots + e_s = 1$, and $\text{Idem}(A)$ is finite. Moreover, $A \cong \bigoplus_{i=1}^s e_i A$, and each $e_i A \cong A/(1 - e_i)A$ is indecomposable.*

Proof. We start with proving that any idempotent is the sum of mutually orthogonal primitive idempotents. Let $e \in \text{Idem}(A)$. If e is not primitive, then we can find non-zero orthogonal idempotents $e_1, e_2 \in A$ such that $e = e_1 + e_2$. By Lemma 11.1(2), we have $e_1, e_2 \leq e$, whence $e_1, e_2 < e$, since e_1, e_2 are non-zero. If either one of e_i is not primitive,

we may continue in this way, and split it up as a sum of two orthogonal idempotents both strictly less than e_i . This cannot go on indefinitely, for then we would get an infinite strictly descending chain in $\text{Idem}(A)$, contradicting Lemma 11.2. This shows that any idempotent admits a decomposition into mutually orthogonal primitive idempotents. In particular,

$$(52) \quad 1 = e_1 + \cdots + e_s,$$

with all e_i mutually orthogonal primitive idempotents. Suppose $1 = f_1 + \cdots + f_t$ is another such decomposition. Multiplying (52) with f_1 , we get

$$(53) \quad f_1 = e_1 f_1 + \cdots + e_s f_1.$$

Since all $e_i f_1$ are orthogonal, and since f_1 is primitive, all terms in the right hand side of (53) must be zero except one, say, $e_1 f_1 = f_1$. In particular, $f_1 \leq e_1$, whence $f_1 = e_1$ by Lemma 11.1(5). This shows that the e_i are unique. Let e be an arbitrary idempotent. Multiplying (52) with e , we get $e = ee_1 + \cdots + ee_s$. Since $ee_i \leq e_i$ and e_i is primitive, Lemma 11.1(5) implies that ee_i is either zero or equal to e_i . In conclusion, e is the sum of some of the e_i (in fact, those such that $e_i \leq e$). In particular, if e is primitive and non-zero, then it must be equal to one of the e_i , showing that e_1, \dots, e_s are all primitive idempotents. Moreover, since any idempotent is a sum of these, $\text{Idem}(A)$ is finite. The last assertion now follows readily. \square

11.4. Theorem. *Two affine X -schemes are stably isomorphic over X if and only if they are isomorphic over X .*

Proof. Let B and \tilde{B} be two finitely generated A -algebras, such that their corresponding affine schemes are stably isomorphic. Hence, there exists a finitely generated A -algebra C such that $D := C \oplus B$ and $\tilde{D} := C \oplus \tilde{B}$ are isomorphic as A -algebras. In fact, all we will use from these rings is that they are Noetherian, and hence we will show that, if $C \oplus B \cong C \oplus \tilde{B}$, for some Noetherian rings B , \tilde{B} , and C , then $B \cong \tilde{B}$. Let c_1, \dots, c_s be the primitive idempotents of C , which are finite in number by Proposition 11.3, and likewise e_1, \dots, e_n and $\tilde{e}_1, \dots, \tilde{e}_{\tilde{n}}$ those of B and \tilde{B} respectively. Hence $(d_1, \dots, d_{s+n}) := (c_1, \dots, c_s, e_1, \dots, e_n)$ and $(\tilde{d}_1, \dots, \tilde{d}_{s+\tilde{n}}) := (c_1, \dots, c_s, \tilde{e}_1, \dots, \tilde{e}_{\tilde{n}})$ are the primitive idempotents of D and \tilde{D} respectively (since their sums are equal to one; see Proposition 11.3).

Since the isomorphism $f: D \rightarrow \tilde{D}$ must preserve primitive idempotents, $n = \tilde{n}$ and there exists a permutation σ of $\{1, \dots, n+s\}$ such that $f(d_i) = \tilde{d}_{\sigma(i)}$. Fix some $1 \leq i \leq s+n$ and let $i' := \sigma(i)$. Since $f(d_i) = \tilde{d}_{i'}$, the isomorphism f induces an isomorphism between $d_i D \cong D/(1-d_i)D$ and $\tilde{D}/(1-\tilde{d}_{i'})\tilde{D} \cong \tilde{d}_{i'}\tilde{D}$. Assume $i' \leq s$, so that $\tilde{d}_{i'}\tilde{D} \cong c_{i'}C \cong d_{i'}D$. In other words, we showed that $d_i D \cong d_{\sigma(i)}D$ whenever $\sigma(i) \leq i$. Letting $i'' := \sigma(i')$, we may repeat this argument, and hence, if $i'' \leq s$, we also get $d_i D \cong d_{\sigma^2(i)}D$. In conclusion, if k is the smallest positive power of σ such that $\sigma^k(i) > s$ (it is possible that no such power exists), then $d_i D \cong d_{\sigma^{k-1}(i)}D \cong \tilde{e}_{\sigma^k(i)-s}\tilde{B}$.

In particular, if $i > s$, then there exists such a smallest power k (since the order of σ is an upper bound for k). Writing $j := i - s$ and $j' := \sigma^k(i) - s$, we showed that for each $j = 1, \dots, n$, there exists $j' \in \{1, \dots, n\}$ such that $e_j B \cong \tilde{e}_{j'}\tilde{B}$. Furthermore, the assignment $j \mapsto j'$ is injective, for if $j'_1 = j'_2$, then for some k_1 and k_2 we have

$$(54) \quad \sigma^{k_1}(j_1 + s) = \sigma^{k_2}(j_2 + s),$$

where k_1 and k_2 are the respective smallest powers of σ that map $j_1 + s$ and $j_2 + s$ to a value greater than s . Without loss of generality, assume $k_2 \leq k_1$. Hence applying σ^{-k_2}

to (54) yields $\sigma^{k_1-k_2}(j_1 + s) = j_2 + s > s$, so that by minimality, $k_1 = k_2$, and hence $j_1 = j_2$. In particular, by the pigeonhole principle, the assignment $j \mapsto j'$ is a bijection, and, therefore, using Proposition 11.3, we get an isomorphism

$$B \cong \bigoplus_{j=1}^n e_j B \cong \bigoplus_{j=1}^n \tilde{e}_{j'} \tilde{B} \cong \tilde{B},$$

as required. \square

REFERENCES

- [1] S. Bosch, W. Lütkebohmert, and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822 (91i:14034) [28](#)
- [2] R. Cluckers and D. Haskell, *Euler characteristic and dimension of semi-algebraic p-adic sets*, preprint, 2000. [6](#)
- [3] ———, *Grothendieck rings of \mathbb{Z} -valued fields*, Bull. Symbolic Logic **7** (2001), no. 2, 262–269. [6](#)
- [4] A. Craw, *An introduction to motivic integration*, Strings and geometry, Clay Math. Proc., vol. 3, Amer. Math. Soc., Providence, RI, 2004, pp. 203–225. MR MR2103724 (2005k:14027) [38](#)
- [5] J. Denef and F. Loeser, *Motivic Igusa zeta functions*, J. Algebraic Geom. **7** (1998), no. 3, 505–537. [36](#)
- [6] ———, *Germes of arcs on singular algebraic varieties and motivic integration*, Invent. Math. **135** (1999), no. 1, 201–232. [28](#), [38](#)
- [7] ———, *On some rational generating series occurring in arithmetic geometry*, Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, pp. 509–526. [36](#), [37](#)
- [8] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. [32](#)
- [9] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977. [39](#)
- [10] Ehud Hrushovski and David Kazhdan, *The value ring of geometric motivic integration, and the Iwahori Hecke algebra of SL_2* , Geom. Funct. Anal. **17** (2008), no. 6, 1924–1967, With an appendix by Nir Avni. [6](#)
- [11] J. Krajíček and T. Scanlon, *Combinatorics with definable sets: Euler characteristics and Grothendieck rings*, Bull. Symbolic Logic **6** (2000), no. 3, 311–330. [6](#)
- [12] E. Looijenga, *Motivic measures*, Astérisque (2002), no. 276, 267–297, Séminaire Bourbaki, Vol. 1999/2000. MR MR1886763 (2003k:14010) [38](#)
- [13] D. Marker, *Model theory*, Graduate Texts in Mathematics, vol. 217, Springer-Verlag, New York, 2002. [20](#)
- [14] H. Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1986. [31](#)
- [15] H. Schoutens, *Existentially closed models of the theory of Artinian local rings*, J. Symbolic Logic **64** (1999), 825–845. [29](#)

DEPARTMENT OF MATHEMATICS, CITY UNIVERSITY OF NEW YORK, 365 FIFTH AVENUE, NEW YORK, NY 10016 (USA)

E-mail address: hschoutens@citytech.cuny.edu